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# Application of Homotopy Perturbation and Reconstruction of Variational Iteration Methods for Harry Dym Equation and Compared with Exact Solution

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***Abstract***

*In this paper, an approximate analytical solution of Harry Dym equation is obtained by using reconstruction of variational iteration method (RVIM) and homotopy perturbation method (HPM), RVIM is based on use of Lagrange multipliers for identification of optimal values of parameters in a function by using Laplace transform. The excellent accuracy of the obtained results is demonstrated by comparing them with available analytical and numerical results in the literature. Both of methods have considerable error for large values of t and x, but the Harry Dym equation is applicable for a limited range of t and x. As a result, RVIM is better than HPM especially for larger values of time parameter and the obtained results are close to exact results. The obtained results expose effectiveness and capability of this method to solve the nonlinear systems in mechanics, analytically.*

***Keywords:*** *Harry Dym equation, Homotopy Perturbation Method, Reconstruction Variational Iteration Method, nonlinear equation, approximate solution.*

## 1. Introduction

The Harry Dym equation is a partial differential equation that was found by Harry Dym in 1973–1974 while trying to transfer some results about isospectral flows to the string equation. But, this equation for the first time published in a 1975 paper of M.D. Kruskal [1]. It arises, e.g., in the analysis of the Saffman–Taylor problem with surface tension [2]. The Harry Dym equation can be written as:

$$u_t = u^3 u_{xxx} \tag{1}$$

The Harry Dym equation represents a system in which scattering and non-linearity are coupled together. The Harry Dym equation has strong links to the KdV equation and applications of this equation were found to the problems of hydrodynamics [3]. The Harry Dym equation is an important dynamical equation and supplies applications in some physical systems, so many researchers have studied on suitable method of solution for this important non-linear equation. The (2+ 1)-dimensional Harry Dym equation is solved by Dmitrieva and Khlabystova [4,5]. Also the well known link between Korteweg-de Vries and Harry Dym equation is improved by Fuchssteiner et al. [6]. The Harry Dym equation is generalized to the system of equations by Popowicz [7]. Additionally, the construction of coupled Harry Dym equation hierarchy [8] and an extended Harry Dym hierarchy [9] are done. A general formula of flow equations for Harry Dym Hierarchy is proposed by Ji-Peng et al. [10] and the exact solution of Harry Dym equation is investigated by Mokhtari [11].

In this paper, new and comfortable methods that have high accuracy, are used to solve Harry Dym equation, reconstruction of variational iteration method (RVIM) and homotopy perturbation method (HPM) are the most effective and convenient ones for both weakly and strongly nonlinear equations. The homotopy perturbation method, proposed first by He in1998, was further developed and improved by him [12,13]. The HPM has been shown to solve large class of nonlinear problems efficiently, accurately and easily, with approximations converging very rapidly to solution. Usually, few iterations lead to high accuracy of the solution. Therefore, this method is very useful and applicable for researchers in engineering sciences, lately [14-18].

The VIM is proposed by the Chinese mathematician Ji-Huan He [19,20], and is a modified general Lagrange's multiplier method. VIM has been favorably applied to various kinds of nonlinear problems. VIM has many merits over classical approximate techniques; it can solve nonlinear equations easily and accurately. In this method, general Lagrange multipliers are introduced to construct correction functional for the problems. The multipliers can be identified optimally via the variational theory. This method has recently been applied to various engineering problems [21-28]. The reconstruction of variational Iteration Method (RVIM) is a powerful and congruous method to be applied to the linear and nonlinear engineering problems. The VIM developed to RVIM where the Laplace transform is used to find Lagrange's multiplier [29-32].

**2. Reconstruction of variational iteration method (RVIM)**

In this section, we briefly review the RVIM to solve a general nonlinear initial value problem. In this method, the problem is initially approximated with possible unknowns. Then, a corrected functional is constructed through a general Lagrange multiplier, which can be identified optimally via the variational theory [33]. To illustrate the basic idea of the method, consider the following general nonlinear system:

$$L[u]+N[u]=g \tag{2}$$

where $L$ is a linear differential operator, $N$ a nonlinear analytic operator, and $g$ an inhomogeneous term.

$$u_{n+1}=u_n+\int_0^t \lambda(L(u_n)+N(\tilde{u}_n)-g)d\tau \tag{3}$$

Where $\lambda$ is a general Lagrangian multiplier which can be identified optimally via the variational theory. The subscript $n$ indicates the $n$th approximation and $\tilde{u}_n$ is considered as a restricted variation, i.e., $\delta\tilde{u}_n=0.$

**3. Implementation of the RVIM in Dym equation**

In order to solve Eq. 1 using RVIM, we construct a correction functional, as follows:

$$u_{n+1}=u_n+\int_0^t \lambda(\frac{\partial u_n}{\partial \tau}-u_n^3\frac{\partial^3 u_n}{\partial x^3})d\tau \tag{4}$$

To obtain $\lambda$, we use Laplace transform. we apply Laplace transform on linear part of equation.

$$\ell(u_t)=(-1)^m \tag{5}$$

where $\ell$ is Laplace operator and $m$ is degree of the highest order derivative of the selected linear operator. In present problem: $m=1$.
So;

$$u=\ell^{-1}(\frac{-1}{s})=-1 \tag{6}$$

$$\lambda=\lim_{t\to t-\tau} u=-1 \tag{7}$$

Then, as a guess, we consider $u_0$ equal to initial condition [10].

$$u_0(x,t)=u(x,0)=(a-\frac{3b}{2}x)^{\frac{2}{3}} \tag{8}$$

Therefore, Eq. 4 is written as follows:

$$u(x,t)=(a-\frac{3b}{2}x)^{\frac{2}{3}}-\int_0^t(\frac{\partial}{\partial\tau}[(a-\frac{3b}{2}x)^{\frac{2}{3}}]-(a-\frac{3b}{2}x)^2\frac{\partial^3}{\partial x^3}[(a-\frac{3b}{2}x)^{\frac{2}{3}}])d\tau \tag{9}$$

**4. Homotopy perturbation method (HPM)**

To illustrate the basic idea of the method, consider the following general nonlinear system:

$$L[u]+N[u]-f=0 \tag{10}$$

where $L$ is a linear differential operator, $N$ a nonlinear analytic operator, and $f$ an inhomogeneous term.
The homotopy perturbation structure is shown as follows:

$$H(u,p)=(1-p)[L(u)-L(u_0)]+p[L(u)+N(u)-f]=0 \tag{11}$$

where $p$ is homotopy parameter $(p\in[0,1])$ and $u_0$ is the first approximation that satisfies the boundary condition. We can assume that the solution of Eq. (5) can be written as a power series in $p$, as follows:

$$u=u_0+pu_1+p^2u_2+... \tag{12}$$

**5. Implementation of the HPM in Dym equation**

To solve Eq. 1 by HPM, we rewrite the Eq. 11:

$$H(u,p)=(1-p)[u_t]+p[u_t-u^3u_{xxx}]=0 \tag{13}$$

By substituting Eq. 12 into the Eq. 13 and rearranging based on power of $p$ terms, we have:

$$p^0:\frac{\partial u_0}{\partial t}=0$$

$$p^1:\frac{\partial u_1}{\partial t}-u_0^3\frac{\partial^3 u_0}{\partial x^3}=0$$

$$p^2:\frac{\partial u_2}{\partial t}-u_0^3\frac{\partial^3 u_1}{\partial x^3}-3u_0^2u_1\frac{\partial^3 u_0}{\partial x}=0$$

By solving the above system of equations, we have:

$$u_1(x,t)=-b^3(a-\frac{3b}{2}x)^{\frac{-1}{3}}t$$

$$u_2(x,t)=-b^6(a-\frac{3b}{2}x)^{\frac{-4}{3}}\frac{t^2}{4}$$

By substituting the various components of $u_n(x,t)$ into the Eq. 12, $u(x,t)$ will be obtained and the Harry Dym equation will be solved by using of HPM.

**6. Results and discussion**

In this section, we have discussed the error analysis between the exact solution and approximate solutions. The exact solution of the Harry Dym equation is as follows [11]:

$$u(x,t)=(a-\frac{3b}{2}(x+t))^{\frac{2}{3}} \tag{14}$$

The absolute error at constants value of $a$=4 and $b$=1, is computed as follows:

$$Error(\%) = \frac{|u(x,t) - \tilde{u}(x,t)|}{u(x,t)} \times 100 \qquad (15)$$

where $\tilde{u}(x,t)$ and $u(x,t)$ are approximate and exact solutions of Eq. 1 respectively.

Numerical illustrations for RVIM, HPM and exact solutions of Harry Dym equation for some values of *x* and *t*, are presented in Table 1 and 2. Also in these tables, error of each analytical solution compared with the exact result.

**Table 1**. Comparing absolute errors (%) for *u(x,t)* given by RVIM and HPM for *x=0.4* and *x=0.8.*

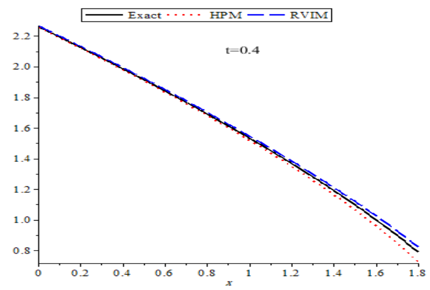
| | *x* =0.4 | | | | | *x* =0.8 | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| *t* (s) | Exact | HPM | RVIM | Error(HPM) | Error(RVIM) | Exact | HPM | RVIM | Error(HPM) | Error(RVIM) |
| 0.2 | 2.126 | 2.125 | 2.128 | 0.04 | 0.09 | 1.842 | 1.8406 | 1.8446 | 0.07 | 0.14 |
| 0.4 | 1.987 | 1.98 | 1.995 | 0.34 | 0.42 | 1.6915 | 1.6808 | 1.7027 | 0.63 | 0.66 |
| 0.6 | 1.842 | 1.819 | 1.862 | 1.24 | 1.08 | 1.534 | 1.4981 | 1.5608 | 2.33 | 1.75 |
| 0.8 | 1.692 | 1.638 | 1.729 | 3.18 | 2.21 | 1.3679 | 1.2838 | 1.4189 | 6.15 | 3.72 |

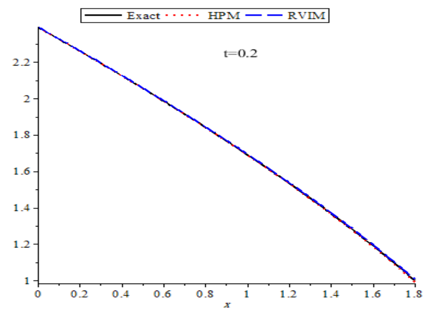
**Table 2.** Comparing absolute errors (%) for *u(x,t)* given by RVIM and HPM for *x=1.2* and *x=1.6.*

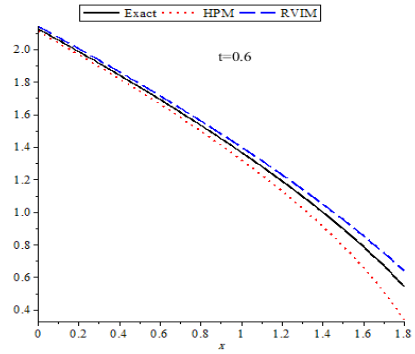
| | *x* =1.2 | | | | | *x* =1.6 | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| *t* (s) | Exact | HPM | RVIM | Error(HPM) | Error(RVIM) | Exact | HPM | RVIM | Error(HPM) | Error(RVIM) |
| 0.2 | 1.534 | 1.532 | 1.534 | 0.15 | 0.24 | 1.1911 | 1.1861 | 1.1969 | 0.41 | 0.49 |
| 0.4 | 1.368 | 1.349 | 1.384 | 1.36 | 1.16 | 1 | 0.9609 | 1.0259 | 3.9 | 2.59 |
| 0.6 | 1.191 | 1.129 | 1.23 | 5.24 | 3.28 | 0.7883 | 0.6596 | 0.8549 | 16.33 | 8.44 |
| 0.8 | 1 | 0.854 | 1.076 | 14.55 | 7.64 | 0.5428 | 0.2493 | 0.6839 | 54.06 | 25.99 |

Variation of *u(x,t)* with *x* is shown in Fig. 1 for several values of *t*, that indicates proximity of approximate results to the exact results at small values of *t*. As can be seen, with increasing *t*, the accuracy of RVIM and HPM results are decreased. Of course, small *t* and *x* are more important for Harry Dym equation, because values of *u(x,t)* will tend towards zero for large values of *t* and *x*.

Variation of absolute percent error for *u(x,t)* given by RVIM and HPM versus x is shown in Fig. 2 for several values of *t*. Fig. 2 shows that the RVIM has higher accuracy than HPM except for small values of *t*.



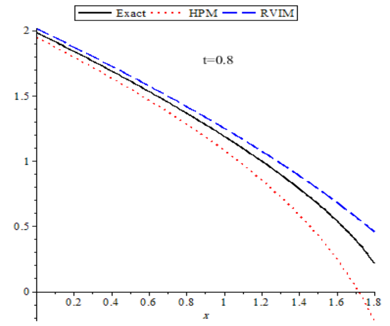
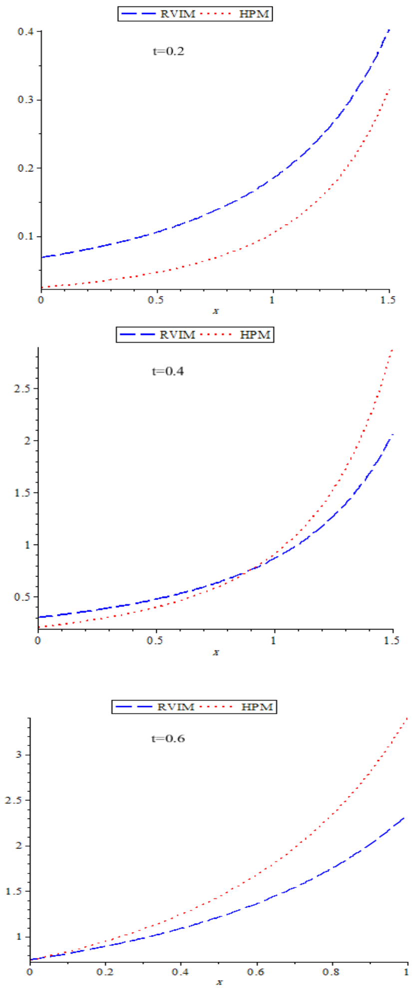




**Figure 1**. Comparison between results of HPM, Exact and RVIM during *x* at *t*=0.2,0.4,0.6,0.8.



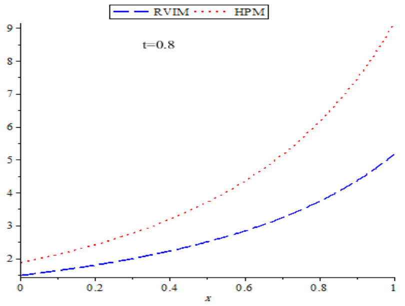



**Figure 2**. Comparison between percent error of HPM and RVIM during *x* at *t*=0.2,0.4,0.6,0.8.

## 7. Conclusions

In present study, the reconstruction of variational iteration method and homotopy perturbation method has been used to obtain an analytical solution to Harry Dym equation, and the results are compared with numerical solution results, that present results are consistent with the high accuracy of the numerical results. The obtained results show that RVIM and HPM are in agreement with exact results, but with increasing *t*, the accuracy of RVIM and HPM results are decreased. The RVIM showed the higher accuracy than the HPM except for small values of *t*. The study led to this conclusion that the RVIM methodology is a very powerful and simple technique for wide classes of problems. In addition, as the RVIM does not require computing Lagrange multiplier and making homotopy multiplier in HPM. Also it is noteworthy to point out that the advantage of the RVIM is the fast convergence of the solutions**.**

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