

## Perihelion Precession in the Solar System

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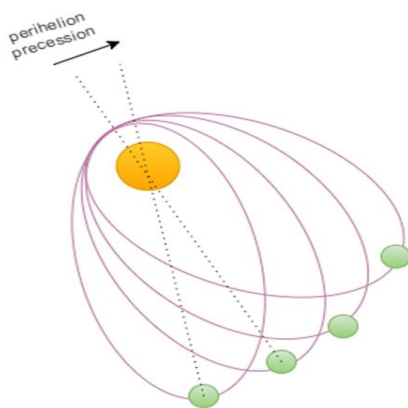
### Abstract

In this paper, we investigate perihelion precession in the solar system, which is one of the most interesting aspects of astrophysics including both aspects of General Relativity and classical mechanics. The phenomenon, where the perihelion of the elliptical orbital path of a planet seems to rotate around a central body (the Sun in this case), is referred to as the precession of the orbital path. Astronomers discovered this natural phenomenon years ago where they failed to explain many strange observatory data. This paper addresses the derivation of the motion equation and the corresponding approximate solution, which results in the perihelion advance formula. We are, therefore, aiming primarily at obtaining solutions to equations of motion and deriving a general formula by taking the General Relativity concepts and Classical Mechanics into account.

**Keywords:** Perihelion, Solar System, Perihelion Precession, Planet, Euler-Lagrange equations.

### 1. Introduction

The anomalous precession of the perihelion of Mercury was among the first phenomena that Einstein's General Theory of Relativity explained [1],[2]. The theory owes its success to the numerical value provided by Einstein for the perihelion precession of Mercury, which was greatly similar to the observation value [3],[4]. This resulted in changes in the apprehension of astronomers and physicists about the concepts of space and time and a different way of viewing the problems [5]. Precession has been defined as a change in the orientation of a rotational planet around the Sun or a central body, as illustrated in Fig. (1), where the semi-major axis rotates around the central body [6].



**Fig.1** Exaggerated view of the perihelion precession of a planet

Four elliptical orbits shifting with respect to one another are displayed in the figure [7]. This shifting or advance is referred to as the advance of the perihelion of the planet or the perihelion precession of the planet. Moreover, the aphelion being the opposite point of the perihelion, the longest distance between the planets and the Sun, advancing at the same angular rate as the perihelion, is shown in Figure (1).

### 2. Literature Review

When Urbain Jean Joseph Le Verrier (1811-1877), a French mathematician, reported the perihelion precession for the first time in 1859, investigation of the solar system appealed to astronomers and theorists more than ever. What attracted Le Verrier's attention to the advance of the perihelion of Mercury was its unusual orbital motion [8]. This was associated with an unknown planet that was never found, which he referred to as Vulcan. The value he obtained for the precession of the perihelion using Newtonian mechanics was 38 arc seconds per century [9]. The results obtained by Le Verrier were advertently corrected in 1895 by Simon Newcomb (1835-1909), a Canadian-American astronomer and mathematician [10], whose theory confirmed Le Verrier's finding about the advance of the perihelion of Mercury. Also following the Newtonian method with a few slight changes in the planetary masses, Newcomb obtained the astounding value of 42.95 arc seconds per century for the advance of Mercury, unbelievably close to the actual value.

An important point is that based on Newton’s law, the planets cannot advance when only the gravitational force between the planet and the Sun is taken into account [11]. 90% of the mass of the solar system, however, concerns the Sun, which demonstrates that the masses of the other planets are negligible as compared to that of the Sun. Furthermore, as the light planets move in the static gravitational field of the Sun, their static gravitational potential can also be neglected.

Albert Einstein’s General Theory of Relativity finally provided explanation for the above natural phenomenon and, therefore, acceptable responses to some inquiries later in 1915 [12].

On November 25, 1915, a paper was published by Einstein based on vacuum field equations [13]. His derivation in the paper was actually mathematically interesting, since the equation of motion was obtained there from the vacuum field equation regardless of Schwarzschild metrics. To obtain the solution to the vacuum field equation, Einstein used an approximation to the spherically symmetric metric, used instead of the Cartesian coordinate system. The approximate metric can be expressed in Polar coordinates

$$(dr)^2 = \left(1 - \frac{2m}{r}\right) (dt)^2 - \left(1 + \frac{2m}{r}\right) (dr)^2 - r^2(d\theta)^2 - r^2 \sin^2(\theta) (d\varphi)^2 \tag{1}$$

where  $m$  and  $r$  denote the mass of the central body and the distance between the planet and the Sun, respectively. Einstein’s approximation for the coefficient of  $(dr)^2$  was related to the real one, expressed soon after that by Schwarzschild, as follows

$$(g_{rr})_E = x(g_{rr})_S \tag{2}$$

where

$$x = 1 - \left(\frac{2m}{r}\right)^2 \tag{3}$$

By estimating the Christoffel symbol as well as using his approximate metric for spherical symmetry, Einstein defined the geodesic equations of motion as

$$x\left(\frac{du}{d\varphi}\right)^2 = \frac{2A}{B^2} + \frac{\alpha}{B^2} u - u^2 + \alpha u^3 \tag{4}$$

where  $u = 1/r$ ,  $\varphi$  is the angular coordinate in the orbital plane, and  $A$ , proportional angular momentum, and  $B$ , concerning energy, are the constants of integration. The precise value of  $x$  is 1 based on the Schwarzschild metric but  $(1 - \alpha^2 u^2)$  according to Einstein’s approximation, which he finally decided to change to one after some calculation. The angular difference  $\Delta\varphi$  was obtained by integrating from Eq. (4). The angular difference was calculated by just accurate and necessary values, and two points were considered as limitation, from the aphelion point to the perihelion point. Einstein obtained the arc length from the perihelion to the aphelion and,

equivalently, that from the aphelion to the next perihelion after he solved the polynomial, just as in Eq. (4). Therefore, the total  $\Delta\varphi$  for one orbit from one perihelion to the next obtains through multiplication of the value by two, and precession per orbit obtains through subtraction of this amount by  $2\pi$ . The result was, therefore, obtained as

$$2\Delta\varphi - 2\pi = \frac{6\pi m}{L} \tag{5}$$

where  $L$  and  $m$  denote the semi-latus rectum of the elliptical orbit (55.4430 million km for Mercury) and the Sun’s mass in geometrical units (1.475 km), respectively. Substitution of the values in Eq. (5) will obtain 0.1034 arc seconds per revolution. As Mercury has 414.9378 revolutions per century, the final result is 42.9195 arc seconds per century, close to the observed value. It should also be noted that Einstein’s result applies not only to circular orbits but actually to any eccentricity.

Eight years after he began to work on his gravitational theory in 1907, he managed to make it help him to obtain the perihelion precession of Mercury.

A few months after Einstein published his paper (on December 22), Karl Schwarzschild (1873-1916), a German physicist and astronomer, successfully found the precise solution to Einstein’s field equation of General Relativity for non-rotating gravitational fields. He first changed Einstein’s first order approximation, to which he found a precise solution, and then introduced only one line element, satisfying four conditions of Einstein. Furthermore, Schwarzschild considered a spherical symmetry around the center by postulating a body exactly at the origin of the coordinate system and assuming the isotropy of space and a static solution (where there is no dependence on time). His line element thus best demonstrated the spherical coordinate as

$$(ds)^2 = \left(1 - \frac{\beta}{R}\right) dt^2 - \left(1 - \frac{\beta}{R}\right)^{-1} dr^2 - R^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{6}$$

where  $\beta = 2GM/c^2$ , ( $c^2 = 1$ ) and  $G$  is Newton’s gravitational constant. Schwarzschild’s equation of orbit remained the same as Einstein’s equation, however. He passed away the next year (on May 11) during World War I, and the Astroied 837 Schwarzschilda was named in his honor. It is not really the case that the planets in the solar system other than Mercury do not have precession. Even planets like the Earth or Venus with almost circular orbits or small eccentricity can display precession. It might appear hard to obtain the precession of such orbits, but this has been made possible and more accurate by modern techniques of measurement and computerized analysis of the values.

Several investigations have been conducted in this regard to obtain a more precise value, and the present paper aims to explore more to provide an exact solution to the second and higher order corrections. See Section 2

for details about all the steps. The geodesic equations given by the Schwarzschild gravitational metric constitute the first issue. The motion of the planets is assumed to be a time-like geodesic in the Schwarzschild metric rotating around the Sun. As clear from the computations for obtaining the perihelion precession equation, both important aspects of physics, i.e., the General Relativity Theory and classical mechanics, are taken into account in all the steps.

A table is provided at the end of Section 2 based on some data and the perihelion precession equation displaying the results for eight solar system planets. The table contains a conversion obtaining two different values for the perihelion advance, expressed as (rad/orbit) and (sec/century), related to one another through

The sidereal period for a planet is defined as its orbital period per year. The orbital period for Mercury, say, is 87.969 days, and every year contains 365 days, 5 hours, 48 minutes, and 46 seconds. The values of sidereal period in a year are obtained through division of the two numbers. Following the same rule for each planet, we will obtain the data in Table (1).

**Table 1** Sidereal Periods of the Planets in the Solar System

Planet	Mercury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune
Sidereal period (year)	0.2408	0.6152	1.000	1.8809	11.865	11.865	83.744	165.912

### 3. Finding of Planet Motion Equation

Nearly spherically symmetric, the Sun has a very small radius as compared to the position of the planets. The spacetime around it may thus be considered to be in the form of the solution to Einstein’s vacuum equations, quite famous as the Schwarzschild spacetime with the line element

$$ds^2 = -\left(1 - \frac{2M_\odot}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M_\odot}{r}\right)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (8)$$

where  $M_\odot$  is the mass of the Sun,  $c$  is the speed of light, and  $G = 1$  is Newton’s gravitational constant. The individual effect of the planets in the solar system on the spacetime is assumed to be negligible for their motion, so every planet moves as a test particle. The following Lagrangian can thus be used for the motion of each planet

$$\mathcal{L} = \frac{1}{2} m g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta \quad (9)$$

where  $m$  denotes the planet mass. It should be noted that

$$g_{\alpha\beta} = \text{diag} \left[ -\left(1 - \frac{2M_\odot}{r}\right), \frac{1}{\left(1 - \frac{2M_\odot}{r}\right)}, r^2, r^2 \sin^2\theta \right] \quad (10)$$

is the metric tensor for the Schwarzschild spacetime. If the metric tensor is now applied in the Lagrangian, the following is obtained

$$\mathcal{L} = \frac{m}{2} \left( -\left(1 - \frac{2M_\odot}{r}\right) \dot{t}^2 + \frac{\dot{r}^2}{\left(1 - \frac{2M_\odot}{r}\right)} + r^2(\dot{\theta}^2 + \sin^2\theta \dot{\phi}^2) \right) \quad (11)$$

where dot denotes taking derivative with respect to the proper time  $\tau$ , which is assumed to be measured by an observer located on the particle. The basic equations of motion are consequently obtained by the Euler-Lagrange equations

$$\frac{d}{d\tau} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} \right) - \frac{\partial \mathcal{L}}{\partial x^\alpha} = 0 \quad (12)$$

It should be noted before giving the explicit form of the equations that the angular momentum of the test particle in a preferable direction  $z$  remains constant, because the spacetime is spherically symmetric. The motion is thus known from the very beginning to be in a 2-dimensional plane. Setting the proper system of coordinates can help choose  $\theta = \frac{\pi}{2}$ , where the equatorial plane is, on which basis, the three Euler-Lagrange equations are obtained by the following equations

$$\frac{d}{d\tau} \left[ \left(1 - \frac{2M_\odot}{r}\right) \dot{t} \right] = 0 \quad (13)$$

$$\frac{d}{d\tau} \left( \frac{\partial \mathcal{L}}{\partial \dot{r}} \right) - \frac{\partial \mathcal{L}}{\partial r} = 0 \quad (14)$$

$$\frac{d}{d\tau} (r^2 \dot{\phi}) = 0 \quad (15)$$

As implied by (13) and (15),

$$\left(1 - \frac{2M_\odot}{r}\right) \dot{t} = E \quad (16)$$

and

$$r^2 \dot{\phi} = \ell \quad (17)$$

where  $E$  and  $\ell$  are two integration constants concerning the energy and angular momentum of the test particle. The four-velocity of the planets must satisfy the following equation as they are moving on a timelike worldline

$$U^\mu U_\mu = -1 \quad (18)$$

where  $U^\mu = \frac{\partial x^\mu}{\partial \tau}$ . The explicit form of Eq. (18) is

$$-\left(1 - \frac{2M_\odot}{r}\right) \dot{t}^2 + \frac{\dot{r}^2}{\left(1 - \frac{2M_\odot}{r}\right)} + r^2 \dot{\phi}^2 = -1 \quad (19)$$

where  $\dot{t} = \frac{\pi}{2} \dot{t}$  and  $\dot{\phi}$  are obtained from Eqs. (16) and (17). The proper equation for the radial coordinate is obtained through substitution in Eq. (19)

$$\dot{r}^2 + \left(1 - \frac{2M_{\odot}}{r}\right) \left(1 + \frac{\ell^2}{r^2}\right) = E^2 \tag{20}$$

Next, we use the chain rule to obtain a differential equation for  $r$  with respect to  $\varphi$ , which gives the following equation from (20).

$$\left(\frac{dr}{d\varphi}\right)^2 + \left(1 - \frac{2M_{\odot}}{r}\right) \left(1 + \frac{\ell^2}{r^2}\right) = E^2 \tag{21}$$

or, more conveniently,

$$\left(\frac{dr}{d\varphi}\right)^2 \frac{\ell^2}{r^4} + \left(1 - \frac{2M_{\odot}}{r}\right) \left(1 + \frac{\ell^2}{r^2}\right) = E^2 \tag{22}$$

We now introduce a new variable  $u = \frac{1}{r}$  as in the Kepler problem. Rewriting the last differential equation obtains

$$\left(\frac{du}{d\varphi}\right)^2 + (1 - 2M_{\odot}u) \left(\frac{1}{\ell^2} + u^2\right) = \frac{E^2}{\ell^2} \tag{23}$$

The above equation is the master first order differential equation that should be solved for  $u(\varphi)$ . Taking the derivative of this equation with respect to  $\varphi$  to get closer to it followed by a rearrangement obtains

$$\frac{d^2u}{d\varphi^2} + u = \frac{M_{\odot}}{\ell^2} + 3M_{\odot}u^2 \tag{24}$$

As compared to the Newtonian planet motion equation, the term added here to the Classical Mechanics as a result of General Relativity (GR) is  $3M_{\odot}u^2$ . The following equation obtains the straightforward solution without GR correction

$$u = \frac{1}{r} = \frac{M_{\odot}}{\ell^2} (1 + e \cos(\varphi - \varphi_0)) \tag{25}$$

where  $e$  denotes the eccentricity of the elliptic orbit of the planet, and the initial phase  $\varphi_0$  is an arbitrary constant that can be set to zero without loss of generality, as the coordinate system can rotate about the symmetry axis. It is unfortunately impossible to analytically solve the master Eq. (23) with the GR correction. The correction to the classical motion is very small, though, as suggested by the nature of the additional term, so a proper method of approximation might yield significantly acceptable results. The GR term, therefore, can be considered as a small perturbation to the classical path of the planets. For this purpose, we consider  $\frac{3M_{\odot}}{\ell^2} = \lambda \ll 1$  and thus expand the planet orbits in terms of  $\lambda$ ; i.e.,

$$u = \sum_{n=0}^{\infty} \lambda^n u_n \tag{26}$$

and

$$u'' = \sum_{n=0}^{\infty} \lambda^n u''_n \tag{27}$$

where the prime stands for taking derivative with respect to  $\varphi$ , and  $u_0$  is the orbit without the GR correction; i.e.,

$$u_0 = \frac{M_{\odot}}{\ell^2} (1 + e \cos\varphi), \tag{28}$$

Substituting the above in the master Eq. (2.16) obtains

$$\sum_{n=0}^{\infty} \lambda^n u''_n + \sum_{n=0}^{\infty} \lambda^n u_n = \frac{M_{\odot}}{\ell^2} + \frac{\ell^2}{M_{\odot}} \lambda (\sum_{n=0}^{\infty} \lambda^n u_n)^2 \tag{29}$$

The Newtonian equation of motion obtained by the following is evaluated in the zeroth order

$$u''_0 + u_0 = \frac{M_{\odot}}{\ell^2} \tag{30}$$

The solution has already been provided in Eq. (28). Generally, the  $n^{\text{th}}$  order equation (where  $n \geq 1$ ) is obtained as follows

$$u''_n + u_n = \frac{\ell^2}{M_{\odot}} \sum_{i=0}^{n-1} u_i u_{n-1-i}. \tag{31}$$

The first order equation, say, turns into another second order differential equation:

$$u''_1 + u_1 = \frac{\ell^2}{M_{\odot}} u_0^2 \tag{32}$$

Since  $u_0^2$  is there at the right hand side, the above equation is nonhomogeneous. Listed below are some of the higher order corrections one can extract from the master Eq.(23), where  $n = 2, 3$  and  $4$ , respectively, as admitted by Eq. (31)

$$u''_2 + u_2 = \frac{\ell^2}{M_{\odot}} (2u_0u_1) \tag{33}$$

$$u''_3 + u_3 = \frac{\ell^2}{M_{\odot}} (2u_0u_2 + u_1^2) \tag{34}$$

$$u''_4 + u_4 = \frac{\ell^2}{M_{\odot}} (2u_0u_3 + 2u_1u_2) \tag{35}$$

Next, we will solve the equation of the first order correction with the following explicit form

$$\frac{d^2u_1}{d\varphi^2} + u_1 = \frac{M_{\odot}}{\ell^2} (1 + e \cos\varphi)^2 \tag{36}$$

The above nonhomogeneous second order ordinary differential equation, which has a constant coefficient, is solved in two distinct parts, including the solution to its homogenous form and the particular solution, both to be discussed in the sequel. The homogenous equation is obtained by

$$\frac{d^2u_1}{d\varphi^2} + u_1 = 0 \tag{37}$$

the solution to which reads

$$u_{1h} = A_1 \sin\varphi + B_1 \cos\varphi \tag{38}$$

where  $A_1$  and  $B_1$  are both integration constants. In order to estimate the particular solution to Eq. (36), we can expand the right-hand side as

$$\frac{d^2 u_1}{d\varphi^2} + u_1 = \mu \left( 1 + \frac{e^2}{2} + 2e\cos\varphi + \frac{e^2}{2} \cos 2\varphi \right) \quad (39)$$

where the settings  $\mu = \frac{M_{\odot}}{\rho^2}$  and  $\cos^2\varphi = (1 + \cos 2\varphi)/2$  hold. The following is considered as the ansatz of the nonhomogeneous second order differential equation with constant coefficients according to the standard solution method

$$u_{1p} = A + (B\sin\varphi + C\cos\varphi)\varphi + D\sin 2\varphi + E\cos 2\varphi \quad (40)$$

where the left and right sides of Eq. (39) are matched to obtain all the constants. Applying the ansatz in Eq. (37) yields

$$A + 2B\cos\varphi - 2C\sin\varphi - 3D\sin 2\varphi - 3E\cos 2\varphi = \mu \left( 1 + \frac{e^2}{2} + 2e\cos\varphi + \frac{e^2}{2} \cos 2\varphi \right) \quad (41)$$

where we obtain  $A = \mu(1 + \frac{e^2}{2})$ ,  $B = \mu e$ ,  $C = 0$ ,  $D = 0$  and  $E = -\mu \frac{e^2}{6}$  after matching the two sides. The particular solution thus reads

$$u_{1p} = \left[ 1 + \frac{e^2}{2} + e\varphi\sin\varphi - \frac{e^2}{6} \cos 2\varphi \right] \quad (42)$$

The full solution is finally calculated by summing the homogenous solution and the particular one as follows

$$u_1 = A_1\sin\varphi + B_1\cos\varphi + \left( \frac{M_{\odot}}{\rho^2} \right) \left[ 1 + \frac{e^2}{2} + e\varphi\sin\varphi - \frac{e^2}{6} \cos 2\varphi \right] \quad (43)$$

We can write the homogeneous solution as

$$u_{1h} = \mathcal{A}\cos(\varphi - \varphi_0) \quad (44)$$

As in the case of  $u_0$ , the initial phase  $\varphi_0$  can be set to zero.

The following equation is used for expression of the orbit of a planet around the Sun up to the first order correction:

$$u = \mu \left( 1 + e\cos\lambda + \lambda \left( 1 + \frac{e^2}{2} + e\varphi\sin\varphi - \frac{e^2}{6} \cos 2\varphi \right) \right) \quad (45)$$

where the term  $\mathcal{A}\cos\varphi$  has been absorbed into the other similar term in  $u_0$ . That is, it is not the homogeneous solution that we are actually seeking. The correction that should be considered is rather the particular solution. Therefore, the following settings are considered for the next step

$$u_0 = \mu(1 + e\cos\varphi) \quad (46)$$

As for the second order correction

$$u_2'' + u_2 = \mu(1 + e\cos\varphi) \left( 1 + \frac{2e^2}{3} + e\varphi\sin\varphi - \frac{e^2}{3} \cos^2\varphi \right) \quad (48)$$

The following is obtained using the standard solution method of the particular solution of second order nonhomogeneous differential equation

$$u_2 = \mu \left\{ -\frac{1}{2}e\varphi^2\cos\varphi - \frac{1}{12}e\varphi(-5e^2 + 8e\cos\varphi - 18)\sin\varphi + \frac{1}{12}e^2\cos^2\varphi(e\cos\varphi - 8) + 2 + \frac{4e^2}{3} \right\} \quad (49)$$

The particular solution to  $n = 3$  equation, obtained as

$$u_3 = \mu \left\{ -\frac{1}{6}e\varphi^3\sin\varphi + \frac{1}{12}e\varphi^2(8e\cos^2\varphi - 4e - (5e^2 + 18)\cos\varphi) + \frac{1}{36}e\varphi\sin\varphi((9e\cos\varphi - 84 - 10e^2)e\cos\varphi + 126 + 45e^2) - \frac{1}{54}e^3\cos^3\varphi(e\cos\varphi - 18) - \frac{2}{27}(4e^2 + 27)e^2\cos^2\varphi + 5 + \frac{e^2(13e^2+108)}{27} \right\} \quad (50)$$

#### 4. Perihelion Precession of the Planets

Theoretically, we can continue to any order of corrections; however, it is hardly the case that this goes beyond the first order for the solar system.

$$u \approx \mu \left( 1 + e\cos\varphi + \lambda \left( 1 + \frac{e^2}{2} + e\varphi\sin\varphi - \frac{e^2}{6} \cos 2\varphi \right) \right) \quad (51)$$

The expression that contains  $e\varphi\sin\varphi$  where  $\varphi$  is large becomes significant even though  $\lambda \ll 1$ , and it can, therefore, be simplified even more to obtain

$$u \approx \mu(1 + e(\cos\varphi + \lambda\varphi\sin\varphi)) \quad (52)$$

As  $\varphi$  increases,  $\lambda \ll \lambda\varphi \ll 1$  can still be considered. This in turn implies  $\lambda\varphi \approx \sin(\lambda\varphi)$  and  $\cos(\lambda\varphi) \approx 1$ . Considering the above point, we can apply this into Eq. (51) to obtain

$$u \approx \mu(1 + e(\cos(\lambda\varphi)\cos\varphi + \sin(\lambda\varphi)\sin\varphi)) \quad (53)$$

which yields the following relation using  $\cos(a)\cos(b) + \sin(a)\sin(b) = \cos(a - b)$

$$u \approx \mu(1 + e\cos((1 - \lambda)\varphi)) \quad (54)$$

It is obviously understood from the above that the motion period is no longer  $2\pi$ , and is obtained instead by

$$(1 - \lambda)\Delta \approx 2\pi \quad (55)$$

where  $\Delta$  is the motion period. Consequently,

$$\Delta \approx \frac{2\pi}{1 - \lambda} \quad (56)$$

Applying  $\frac{1}{1 - \lambda} = \sum_{k=0}^{\infty} \lambda^k$ , as  $\lambda \ll 1$ , we obtain

$$\Delta \approx 2\pi(1 + \lambda) \quad (57)$$

in the first order, which demonstrates a perihelion precession per orbit for Mercury because of the GR term equal to  $\delta\Delta = \Delta - \Delta_0 \approx 2\pi\lambda$ , where  $\Delta_0 = 2\pi$  is the period of the orbit of the planet predicted by Newton's gravity.

Since both  $M_\odot$  and  $\ell$  are in natural units,  $\lambda = \frac{3M_\odot^2}{\ell^2}$  is also obtained in natural units, so it has to be converted into geometrized units as follows

$$\lambda = \left(\frac{3M_\odot^2}{\ell^2}\right)_{\text{natural units}} = \left(\frac{3M_\odot^2 G^2}{\ell^2 c^2}\right)_{\text{geometrized units}} \quad (58)$$

Furthermore, the proper coefficients must be used for conversion of mass and angular momentum per unit mass from natural units to geometrized units. Here,  $(M_\odot)_{\text{NU}} = \frac{G}{c^2}(M_\odot)_{\text{GU}}$  and  $(L)_{\text{NU}} = \frac{G}{c^3}(L)_{\text{GU}}$ , which obtains  $(\ell)_{\text{NU}} = \frac{1}{c}(\ell)_{\text{GU}}$ , and in SI units,

$$\delta\Delta \approx \frac{6\pi M_\odot^2 G^2}{\ell^2 c^2} \quad (59)$$

where  $M_\odot$  is the mass of the Sun in kg,  $G$  is Newton's gravitational constant,  $\ell = \frac{L}{m}$ , where  $L$  and  $m$  are the angular momentum and mass of the planet, respectively, and  $c$  is the speed of light in m/s. The following more precise relation, therefore, obtains

$$\delta\Delta \approx \frac{6\pi M_\odot^2 m^2 G^2}{L^2 c^2} \quad (60)$$

Back to the classical Newtonian gravity and the popular Kepler's law, according to the first law, the planets orbit the Sun on an ellipse with the semi-major and semi-minor,  $a$  and  $b$ , respectively, and it should be noted that the Sun is located on one of the foci of the ellipse. Based on the second law, a line from the Sun to the planets sweeps out an equal area in equal time. Finally, the third law suggests that the square of the planet period is proportional to the cube of the semi-major axis. Based on the second and the third laws,

$$T^2 = \frac{4\pi^2(1-e^2)a^4}{\ell^2} \quad (61)$$

And

$$T^2 = \frac{4\pi^2 a^3}{G(M_\odot + m)} \quad (62)$$

which can be approximated as follows, as  $m \ll M_\odot$  for all the planets in our solar system:

$$T^2 \approx \frac{4\pi^2 a^3}{GM_\odot} \quad (63)$$

After substituting in E

$$\delta\Delta = \frac{24\pi^3 a^2}{T^2(1-e^2)c^2} \quad (64)$$

Perihelion precession  $\delta\Delta$  is provided in Table 2 for all the planets in the solar system.

**Table 2** Perihelion precession of the solar system as affected by general relativity

Planets	a(m) $\times 10^{11}$	T(d)	e	$\delta\Delta\left(\frac{\text{rad}}{\text{orbit}}\right)$ $\times 10^{-6}$	$\delta\Delta\left(\frac{\text{sec}}{\text{century}}\right)$
Mercury	0.57909	87.969	0.205630	0.5018545204	42.980
Venus	1.08208	224.70	0.006773	0.2571130671	8.6247
Earth	1.49597	365.25	0.016710	0.1859498484	3.8374
Mars	2.27936	686.98	0.093412	0.1230815591	1.3504
Jupiter	7.78412	4332.5	0.048392	0.0358103619	0.0623
Saturn	14.2672	10759.	0.054150	0.0195494684	0.0136
Uranus	28.7097	30685	0.047167	0.0098226489	0.0024
Neptune	44.9825	60189	0.008585	0.0061828420	0.0008

### Conclusion

We have investigated "Perihelion Precession in the Solar System" in this paper. It is presented a review from the initial idea of the issue to the final correction made to the equations for obtaining the best values. It was explained that the idea was initially suggested by the Newtonian law, and some astronomers and mathematicians attempted to provide methods with results closest to the observed value. Perihelion precession was among the phenomena solved by the theory. Beginning with Schwarzschild space time equation in the second section, we assumed for the motion of the planets in the solar system that every planet moves as a test particle. Solving the Euler-Lagrangian equations provides the best way for deriving the equation of motion for planets. The line element is given here in Schwarzschild coordinates. We obtained the two conserved quantities energy and angular momentum of the test particle. We obtained a general equation in order to get the higher order.

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