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# Dual-Phase-Lagging Thermoelastic Damping Vibration in Micro- Nano Scale Beam Resonators with Voids

**N. A. Alghamdi**

Mathematics Dept., Faculty of Sci., Umm Al-Qura Uni., Makkah, KSA

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***Abstract***

*In this paper, the Thermoelastic damping vibration with voids of beam resonator is analyzed by dual-phase-lagging generalized thermoelasticity theory. The basic equations and the boundary-value problems are formulated. The Q-factor for thermoelastic damping has been derived and the effects of voids have been discussed. Analytical expressions for deflection, temperature change, frequency shifts and Thermoelastic damping in the beam have been derived.*

***Keywords:*** *Dual-Phase-Lagging; Thermoelastic Damping; Vibration of beam resonators; voids.*

## Introduction

Micro-and nanomechanical beam resonators are a very important issue for physical applications (1-4). Also, a lot of scientists have interest with Micro- or nanoelectromechanical in recent years. Quality factor or Q factor is an important parameter of a micro-resonator. Higher Q indicates a lower rate of energy loss relative to the stored energy of the resonator. Resonators with high-quality factors have low damping so that they ring or vibrate longer. Therefore, the study of energy dissipation mechanism is of great importance for development and improvement of the planning of micro/ nanoelectromechanical resonators (5).

Sharma and Grover (6) studied Thermoelastic vibrations in micro-/nano-scale beam resonators with voids. Sun and Saka (7) inspected the thermoelastic damping vibration in circular resonators. Their formula of Thermoelastic damping for microplate resonators is different from that of Lifshitz and Roukes (8) by a factor $K=(1+\upsilon)/(1-2\upsilon)$, in which $\upsilon$ is Poisson's ratio. Sharma and Sharma (9) studied thermoelastic damping in micro-scale circular plate resonators. They employed the generalized theory of thermoelasticity of Lord and Shulman's model. Some searches compute the magnitude of TED in the beam (6), plates (10, 11) and ring (12). Sun and Tohmyoh (13) studied thermoelastic damping of the axisymmetric vibration in circular beams.

The dual-phase-lagging (DPL) model has been used for studying the lagging response in conductive heat transfer at the microscale (14, 15). Various heat transfer problems have been described with the DPL model. The physical meanings of the DPL model are shown by the experimental results (15). Xu (16) studied a heat transfer problems. Also, Guo *et al*. (17) analyzed thermoelastic damping in micro- and nanomechanical resonators based on dual-phase-lagging generalized thermoelasticity theory. Tzuo (14) had introduced another modification to Fourier law, he proposed a form of the energy equations, he introduced the two lags, the heat flux time lag, and the temperature gradient time lag. Therefore he had used the dual phase lag heat convection equation with the energy conservation law to obtain the dual phase lag model for heat convection.

Al-Nimr and Al-Huniti (18) studied thermal stresses of thin plate induced by a rapid heating. Ho *et al*. (19) analyzed the heat transfer and the transmission–reflection phenomenon on the surface of a two-layered structure.

The theory of elastic materials with voids is concerned with the elastic materials consisting of a distribution of small pores (voids) containing nothing of mechanical or energetic significance and is one of the most recent generalizations of the classical theory of elasticity.

Nunziato and Cowin (20) presented a nonlinear theory of elastic materials with voids. Cowin and Nunziato (21) presented a linear theory of elastic materials with voids. Puri and Cowin (22) studied the behavior of plane harmonic waves in linear theory of elastic materials with voids. Now this theory has been extended and applied in various types of materials such as viscoelastic, micropolar and thermoelastic to solve problems of interest. Iesan (23) developed a linear theory of thermoelastic materials with voids. Sharma and Kaur (24) studied the plane harmonic waves in generalized thermoelastic materials with voids. Sharma *et al*. (25) carried out an exact free vibration analysis of simply supported, homogenous isotropic, cylindrical panel in the three-dimensional generalized thermoelasticity with voids. Sharma and Grover (6) studied thermoelastic vibrations in micro/nano-scale beam resonators with voids.

The purpose of the present work is to study the voids and thermal relaxation time, based on the dual phase lag modification of Non – Fourier beam resonators. The problem is formulated in the dimensionless form and then solved analytically. The displacement and the temperature solutions are obtained. An exact free vibration analysis of simply supported, homogeneous isotropic are presented.

**Mathematical equations**

We consider small flexural deflection of a thin thermoelastic beam with voids of length L and a rectangular cross-section of dimensions $h\times a$. Take x-axis along the axis of the beam, y-axis along the thickness and z-axis along the width direction. In equilibrium, the beam is unstrained, unstressed and also kept at uniform temperature $T_0$ and volume fraction $\phi_0$.

Thus, the displacements, strains and stresses as:

$$u_x = -y\frac{\partial w(x,t)}{\partial x},\ u_y = 0,\quad u_z = w(x,t), \tag{1}$$

$$e = e_{xx} + e_{yy} + e_{zz} = -y\frac{\partial^2 w(x,t)}{\partial x^2}, \tag{2}$$

$$\sigma_{ij} = \lambda\delta_{ij}e_{kk} + 2\mu e_{ij} + \left(b\phi - \beta T\right)\delta_{ij},\quad i,j,k = 1,2,3, \tag{3}$$

where w, t are the deflection of the beam and the time respectively.

$\lambda$, $\mu$ are the Lame's parameters; $\beta = (3\lambda + 3\mu)\alpha_T$, $\alpha_T$ is the linear thermal expansion. The flexural moment of the cross-section is given as follows:

$$M(x,t) = \int_{-h/2}^{h/2} a\sigma_{xx} y\,dy = I\left(\lambda + 2\mu\right)\frac{\partial^2 w}{\partial x^2} - bM_\phi + \beta M_T, \tag{4}$$

where $I = \frac{ah^3}{12}$ is a moment of inertia of the beam and

$$M_\phi = a\int_{-h/2}^{h/2} \phi y\,dy \quad \text{and} \quad M_T = a\int_{-h/2}^{h/2} Ty\,dy, \tag{5}$$

are the moment of the beam due to the presence of voids and thermal effects, respectively.

For this case, the equation of motion with Thermoelastic coupling for the beam is given by

$$\frac{\partial^2 M}{\partial x^2} + \rho A\frac{\partial^2 w}{\partial t^2} = \Delta P, \tag{6}$$

where $\rho$ is the density; $A = h\times a$ is the area of cross-section and $\Delta P = P_1 - P_2$. Here $P_1$ and $P_2$ the pressures on the upper and lower surface of the beam. Using equation (4) in equation (6) we obtain

$$\left(\lambda + 2\mu\right)I\frac{\partial^4 w}{\partial x^4} - b\frac{\partial^2 M_\phi}{\partial x^2} + \beta\frac{\partial^2 M_T}{\partial x^2} + \rho A\frac{\partial^2 w}{\partial t^2} = \Delta P \tag{7}$$

The equation of balance of equilibrated force is given by(24)

$$\rho\chi\ddot{\phi} = \alpha\nabla^2\phi - b\nabla.\vec{u} - \xi_1\phi - \xi_2\dot{\phi} + mT, \tag{8}$$

where $\vec{u}(x,y,z) = (u_x, u_y, u_z)$ is a displacement vector and $\alpha$, $b$, $\xi_1$, $\xi_2$, $m$ and $\chi$ are the material constants due to voids.

We can write Eqn. (8) as:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} - \frac{\rho\chi}{\alpha}\ddot{\phi} + \frac{by}{\alpha}\frac{\partial^2 w}{\partial x^2} - \frac{\xi_1}{\alpha}\left(\phi + \bar{\xi}\dot{\phi}\right) + \frac{m}{\alpha}T = 0 \tag{9}$$

where $\bar{\xi} = \frac{\xi_2}{\xi_1}$.

The heat conduction equation for a Thermoelastic isotropic body in the context of dual-phase-lagging model with voids is given as (24):

$$K\nabla^2 T - \rho C_\upsilon\left(\dot{T} + \tau_q\ddot{T}\right) = \beta T_0\left(\nabla.\dot{\vec{u}} + \tau_q\nabla.\ddot{\vec{u}}\right) + mT_0\left(\dot{\phi} + \tau_q\ddot{\phi}\right) \tag{10}$$

We can rewrite the above equation in the form

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} - \frac{\rho C_\upsilon}{K}\left(\dot{T} + \tau_q\ddot{T}\right) - \frac{mT_0}{K}\left(\dot{\phi} + \tau_q\ddot{\phi}\right) + \frac{\beta T_0}{K}y\frac{\partial^2}{\partial x^2}\left(\dot{w} + \tau_q\ddot{w}\right) + \tau_T\left(\frac{\partial^3 T}{\partial x^2\partial t} + \frac{\partial^3 T}{\partial y^2\partial t}\right) = 0 \tag{11}$$

where $K$ is the thermal conductivity of the material, $\rho$ and $C_\upsilon$ are the density and specific heat at constant strain, respectively, $\tau_q$ and $\tau_T$ are the heat flux and the phase lags of the temperature gradient vector, respectively.

Here the system of Eqs. (7, 9, 11) governs the transverse vibrations in a Thermoelastic beam with voids. In order to solve the last system, we can assume that:

$$w(x,t) = W(x)e^{i\omega t},\quad \phi(x,y,t) = \Phi(x,y)e^{i\omega t}\ T(x,y,t) = \Theta(x,y)e^{i\omega t} \tag{12}$$

Using equation (12) in system of equations (7, 9, 11), we obtain

$$I\left(\lambda + 2\mu\right)\frac{d^4 W}{dx^4} - b\frac{d^2 M_{\phi_0}}{dx^2} + \beta\frac{d^2 M_{T_0}}{dx^2} - \rho A\omega^2 W(x) = 0, \tag{13}$$

$$\frac{\partial^2\Phi}{\partial x^2} = -\frac{\partial^2\Phi}{\partial y^2} + \left(\frac{\xi_1\bar{\xi}_0 - \rho\chi\omega^2}{\alpha}\right)\Phi - \frac{by}{\alpha}\frac{d^2 W}{dx^2} - \frac{m}{\alpha}\Theta, \tag{14}$$

$$\left(\frac{\partial^2\Theta}{\partial x^2} + \frac{\partial^2\Theta}{\partial y^2}\right)\bar{\tau}_T = \frac{i\omega\rho C_\upsilon}{K}\bar{\tau}_q\Theta + \frac{i\omega m T_0}{K}\bar{\tau}_q\Phi - \frac{i\omega\beta T_0}{K}\bar{\tau}_q y\frac{\partial^2 W}{\partial x^2}, \tag{15}$$

where we considered $\Delta P = 0$, $m = 0$, $\bar{\xi} = \frac{\xi_2}{\xi_1}$,

$\bar{\xi}_0 = 1 + i\omega\bar{\xi}$, $\bar{\tau}_T = 1 + i\omega\tau_T$, $\bar{\tau}_q = 1 + \omega\tau_q$

and

$$M_{\phi_0} = a\int_{-h/2}^{h/2}\Phi y\,dy,\quad M_{T_0} = a\int_{-h/2}^{h/2}\Theta y\,dy. \tag{16}$$

It is assumed that the boundary of the beam are adiabatic, so that

$$\frac{\partial \Phi}{\partial y}=\frac{\partial \Theta}{\partial y}=0 \quad \text{at} \quad y=\pm\frac{h}{2}. \tag{17}$$

In case equation (14-15) are uncoupled to each other in respect of volume fraction $\Phi$ and temperature $\Theta$ so that

$$\Phi(x,y)=\frac{b}{\alpha}\frac{1}{\left(\xi_1\overline{\xi}_0-\rho\chi\omega^2\right)}\left[y-\frac{\sin(qy)}{q\cos(qh/2)}\right]\frac{d^2W}{dx^2}, \tag{18}$$

and

$$\Theta(x,y)=\frac{\beta T_0}{\rho C_\upsilon}\left[y-\frac{\sin(py)}{p\cos(ph/2)}\right]\frac{d^2W}{dx^2}. \tag{19}$$

Differentiating solutions (18-19) w.r.t. $y$ twice and then substituting for $\frac{\partial^2\Phi}{\partial y^2}$ and $\frac{\partial^2\Theta}{\partial y^2}$ in equations (14-15), we get

$$\frac{\partial^2\Phi}{\partial x^2}=-\left[\frac{q^*b}{\alpha\left(\xi_1\overline{\xi}_0-\rho\chi\omega^2\right)}\left(\frac{\sin(qy)}{q\cos(qh/2)}\right)+\frac{m\beta T_0}{\alpha\rho C_\upsilon}\left(y-\frac{\sin(py)}{p\cos(ph/2)}\right)\right]\frac{d^2W}{dx^2} \tag{20}$$

and

$$\frac{\partial^2\Theta}{\partial x^2}=\left[\frac{i\omega m T_0 b}{\alpha K\left(\xi_1\overline{\xi}_0-\rho\chi\omega^2\right)}\frac{\overline{\tau}_q}{\overline{\tau}_T}\left(y-\frac{\sin(qy)}{q\cos(qh/2)}\right)-\frac{p^*\beta T_0}{\rho C_\upsilon}\left(\frac{\sin(py)}{p\cos(ph/2)}\right)\right]\frac{d^2W}{dx^2} \tag{21}$$

where $q^*=q^2+\frac{\left(\xi_1\overline{\xi}_0-\rho\chi\omega^2\right)}{\alpha}$, $p^*=p^2+i\omega\frac{\overline{\tau}_q}{\overline{\tau}_T}\frac{\rho C_\upsilon}{K}$.

Differentiating equation (16) w.r.t. $x$ twice and then substituting for equations (20-21) into $\frac{d^2M_{\phi_0}}{dx^2}$ and $\frac{d^2M_{T_0}}{dx^2}$, we get

$$\frac{d^2M_{\phi_0}}{dx^2}=a\int_{-h/2}^{h/2}\frac{d^2\Phi}{dx^2}y\,dy$$

$$=I\left[\frac{q^*b}{\alpha\left(\xi_1\overline{\xi}_0-\rho\chi\omega^2\right)}g(\omega)-\frac{m\beta T_0}{\alpha\rho C_\upsilon}(1+f(\omega))\right]\frac{d^2W}{dx^2}, \tag{22}$$

and

$$\frac{d^2M_{T_0}}{dx^2}=a\int_{-h/2}^{h/2}\frac{d^2\Theta}{dx^2}y\,dy$$

$$=I\left[\frac{p^*\beta T_0}{\rho C_\upsilon}f(\omega)+\frac{i\omega m T_0 b}{\alpha K\left(\xi_1\overline{\xi}_0-\rho\chi\omega^2\right)}\frac{\overline{\tau}_q}{\overline{\tau}_T}(1+g(\omega))\right]\frac{d^2W}{dx^2}, \tag{23}$$

where

$$g(\omega)=\frac{24}{q^3h^3}\left[\frac{qh}{2}-\tan\left(\frac{qh}{2}\right)\right],$$

$$f(\omega)=\frac{24}{p^3h^3}\left[\frac{ph}{2}-\tan\left(\frac{ph}{2}\right)\right]. \tag{24}$$

Using equations (22-23) in equation (13), we get

$$I(\lambda+2\mu)\frac{d^4W}{dx^4}+\left[bG(\omega)+\beta F(\omega)\right]I\frac{d^2W}{dx^2}-\rho A\omega^2W(x)=0, \tag{25}$$

where

$$G(\omega)=-\frac{q^*b}{\alpha\left(\xi_1\overline{\xi}_0-\rho\chi\omega^2\right)}g(\omega)+\frac{m\beta T_0}{\alpha\rho C_\upsilon}(1+f(\omega)), \tag{26}$$

and

$$F(\omega)=\frac{p^*\beta T_0}{\rho C_\upsilon}f(\omega)+\frac{i\omega m T_0 b}{\alpha K\left(\xi_1\overline{\xi}_0-\rho\chi\omega^2\right)}\frac{\overline{\tau}_q}{\overline{\tau}_T}(1+g(\omega)). \tag{27}$$

Also, using equations (18-19) in (16), then differentiate w.r.t. $x$ twice we get

$$\frac{d^2M_{\phi_0}}{dx^2}=\frac{bI}{\alpha\left(\xi_1\overline{\xi}_0-\rho\chi\omega^2\right)}\left[1+g(\omega)\right]\frac{d^2}{dx^2}\left(\frac{d^2W}{dx^2}\right), \tag{28}$$

and

$$\frac{d^2M_{T_0}}{dx^2}=\frac{I\beta T_0}{\rho C_\upsilon}\left[1+f(\omega)\right]\frac{d^2}{dx^2}\left(\frac{d^2W}{dx^2}\right). \tag{29}$$

The comparison of equations (22-23) with (28-29), we get

$$G(\omega)\cong-\frac{b}{\alpha\left(\xi_1\overline{\xi}_0-\rho\chi\omega^2\right)}\left[1+g(\omega)\right]\frac{d^2}{dx^2}, \tag{30}$$

and

$$F(\omega)\cong\frac{\beta T_0}{\rho C_\upsilon}\left[1+f(\omega)\right]\frac{d^2}{dx^2}. \tag{31}$$

Using equation (30-31) in equation (25), we get

$$D_\omega\frac{d^4W}{dx^4}-\rho A\omega^2W(x)=0, \tag{32}$$

where

$$D_\omega=(\lambda+2\mu)I\left[1-\varepsilon_\phi\left(\overline{\xi}_0-\frac{\rho\chi\omega^2}{\xi_1}\right)^{-1}(1+g(\omega))+\varepsilon_T(1+f(\omega))\right] \tag{33}$$

$$\varepsilon_\phi=\frac{b^2}{\alpha\xi_1(\lambda+2\mu)} \quad \text{and} \quad \varepsilon_T=\frac{\beta^2T_0}{\rho C_\upsilon(\lambda+2\mu)},$$

where $\varepsilon_\phi$ and $\varepsilon_T$ are the elasto-voids and thermomechanical coupling constants of the beam, respectively. In usual, the thermal gradients in the plane of the cross-section along the thickness direction of the beam are much larger than those along the perpendicular to it (7), so that $\partial^2\Theta/\partial x^2=0$. On the same analogy, we assume that the gradient of volume-fraction field is negligible small along perpendicular to the thickness direction of the beam and hence we may take $\partial^2\Phi/\partial x^2=0$. Under these assumptions, we have

$$p^2=-\frac{i\omega\overline{\tau}_q\rho C_\upsilon}{K\overline{\tau}_T}\left[1+\frac{mb}{\beta\left(\rho\chi\omega^2-\xi_1\overline{\xi}_0\right)}\right] \tag{34}$$

$$q^2=\left(\frac{\rho\chi\omega^2-\xi_1\overline{\xi}_0}{\alpha}\right)\left[1+\frac{m\beta T_0}{b\rho C_\upsilon}\right] \tag{35}$$

The solution is given by eq. (18) and (19) with modified values of $p$ and $q$ given by Eqns. (34-35). Thus Eqns. (18), (19) and (32) constitute a complete set of the governing equations for the homogeneous isotropic thin Thermoelastic beam with voids, when there is no pressure difference at the surface occurs. In addition, theses equations can also be supplemented with appropriate initial and boundary conditions of the relevant problem to be modeled.

Now, we consider the case of a micro-beam and nano-beam whose edges are clamped, so that we have the following set of boundary conditions:

$$W=0,\ \frac{dW}{dx}=0 \ \text{ at } \ x=0,L\ . \tag{36}$$

Sense $W=W(x)$, so Eq. (32) becomes as

$$\left[\frac{d^4}{dx^4}-\eta^4\right]W(x)=0, \qquad \eta^4=\frac{\rho A\omega^2}{D_\omega} \tag{37}$$

The solution of Eq. (37) can be written as

$$W(x)=A_1\sin\eta x+A_2\cos\eta x+A_3\sinh\eta x+A_4\cosh\eta x,$$

$$\eta^2=\omega\sqrt{\frac{\rho A}{D_\omega}}\ . \tag{38}$$

Deferent Eq. (38) four times and using boundary condition in Eq. (36), we have

$$\cos\eta L\cosh\eta L=1, \tag{39}$$

where roots of Eq. (39) as

$$\eta=\frac{2k\pi}{L},\quad k\in I\ .$$

Then the deflection can be written with the help of Eq. (12) as

$$w(x,t)=\sum_{k=1}^{\infty}\bar{A}_k\left[\sin\frac{2k\pi}{L}x-\sinh\frac{2k\pi}{L}x\right]e^{i\omega t}, \tag{41}$$

where the vibration frequency of the beam with voids is given by

$$\omega_k=\frac{4k^2\pi^2}{L^2}\sqrt{\frac{D_\omega}{\rho A}}\ . \tag{42}$$

The volume fraction and temperature distributions in the beam are given by:

$$\phi(x,y,t)=-\frac{b}{\alpha}\frac{1}{\left(\xi_1\bar{\xi}_0-\rho\chi\omega^2\right)}\left[y-\frac{\sin(qy)}{q\cos(qh/2)}\right]\sum_{k=1}^{\infty}\bar{A}_k\frac{4k^2\pi^2}{L^2}\left[\sin\frac{2k\pi}{L}x+\sinh\frac{2k\pi}{L}x\right]e^{i\omega t}, \tag{43}$$

$$T(x,y,t)=-\frac{\beta T_0}{\rho C_\upsilon}\left[y-\frac{\sin(py)}{p\cos(ph/2)}\right]\sum_{k=1}^{\infty}\bar{A}_k\frac{4k^2\pi^2}{L^2}\left[\sin\frac{2k\pi}{L}x+\sinh\frac{2k\pi}{L}x\right]e^{i\omega t}, \tag{44}$$

Now, we can write Eq. (42) as

$$\omega_k=\omega_0\sqrt{1-\varepsilon_\phi\left(1+\frac{\rho\chi}{\xi_1}\omega^2\right)\left[1+g(\omega)\right]+\varepsilon_T\left[1+f(\omega)\right]} \tag{45}$$

where

$$\omega_0=\frac{4k^2\pi^2}{L^2}\frac{h}{2}\sqrt{\frac{\lambda+2\mu}{3\rho}}\ .$$

For most of the materials, we can replace $f(\omega)$ and $g(\omega)$ with $f(\omega_0)$ and $g(\omega_0)$, respectively, and expand Eq.(45) up to first order to obtain

$$\omega_k=\omega_0\left[1-\frac{\varepsilon_\phi}{2}\left(1+\frac{\rho\chi}{\xi_1}\omega_0^2\right)\left[1+g(\omega_0)\right]+\frac{\varepsilon_T}{2}\left[1+f(\omega_0)\right]\right]. \tag{47}$$

Clearly the quantity $q$ and $p$ given by Eq. (34-35) are complex, therefore using Euler theorem and replacing $\omega_k$ with $\omega_0$ we obtain

$$p=\sqrt{2}p_0\left(\cos\left(\frac{\theta_1+\theta_2}{2}\right)+i\sin\left(\frac{\theta_1+\theta_2}{2}\right)\right), \tag{48}$$

$$q=\sqrt{2}q_0\left(\cos\left(\frac{\theta_3}{2}\right)+i\sin\left(\frac{\theta_3}{2}\right)\right), \tag{49}$$

where $p_0, q_0, \theta_1, \theta_2$ and $\theta_3$ are defined in Appendix.

Using Eqs. (A.1)- (A.3) and (A.8) in Eq. (47) and simplifying, we obtain (40)

$$\omega_k=\omega_k^R+i\omega_k^I,$$

where

$$\omega_k^I=-\frac{\omega_0}{2}\left[-\varepsilon_\phi\left(1+\frac{\rho\chi}{\xi_1}\omega_0^2\right)H_3+\varepsilon_T H_4\right]\ . \tag{51}$$

where $H_1, H_2, H_3, H_4, \kappa, \hat{\kappa}, \kappa', \hat{\kappa}', T^*$ and $\hat{T}^*$ are defined in Appendix in Eqs. (A.4)- (A.9).

The Thermoelastic damping (TED) and frequency shift are given by

$$Q^{-1}=2\left|\frac{\omega_k^I}{\omega_k^R}\right|=\left|\varepsilon_\phi\left(1+\frac{\rho\chi}{\xi_1}\omega_0^2\right)H_3-\varepsilon_T H_4\right|, \tag{52}$$

and

$$\Omega=\left|\frac{\omega_k^R-\omega_0}{\omega_0}\right|=\left|-\frac{\varepsilon_\phi}{2}\left(1+\frac{\rho\chi}{\xi_1}\omega_0^2\right)H_1+\frac{\varepsilon_T}{2}H_2\right|\ . \tag{53}$$

**Numerical results and discussion**

The theoretical results obtained in the previous section of this article are employed in this part to the discussion of the damping factor $Q^{-1}$ and frequency shift $\Omega$ on the beam dimensions, boundary conditions, vibration modes, environmental temperature, thermal relaxation time and voids for magnesium. The physical constant of magnesium is given in Table 1. The dimensions of the beam and parameters of resonance frequencies of fundamental mode (thickness-shear mode) have been taken in the prescribe limits [9] for micro-scale beam resonators.

In figures 1-4, we represent the variation of thermoelastic damping (scaled by the relaxation strength $\Delta_E$) with varying thickness $\left(10^{-9} \le h \le 10^{-4}\right)$ and fixed length $L = 500\mu m$, for the set I and set II. And varying the thickness and fixed length $L = 100\mu m$, for the set I and set II, respectively. It is observed that the damping factor increases first then decreases in the considered range of thickness (h). Also, It is observed the value of damping factor in curve 2 is greatest value than curve 1 and 3, that's mean a lower rate of energy loss relative to the stored energy of the resonator in curve 2, this Resonator with high-quality factors have low damping so that they ring or vibrate longer.

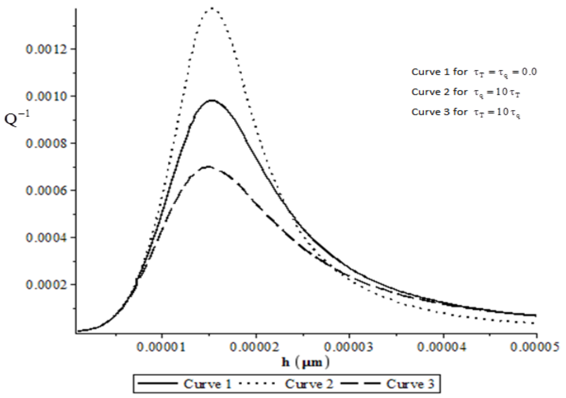


**Figure1:** The thermoelastic damping of clamped beam with length $L = 500\mu m$ for set I

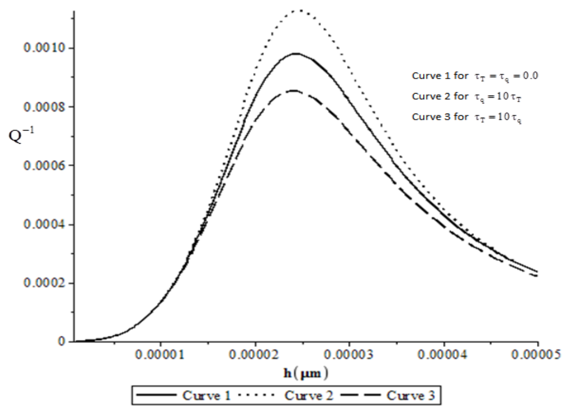


**Figure 2:** The thermoelastic damping of clamped beam with length $L = 500\mu m$ for set II

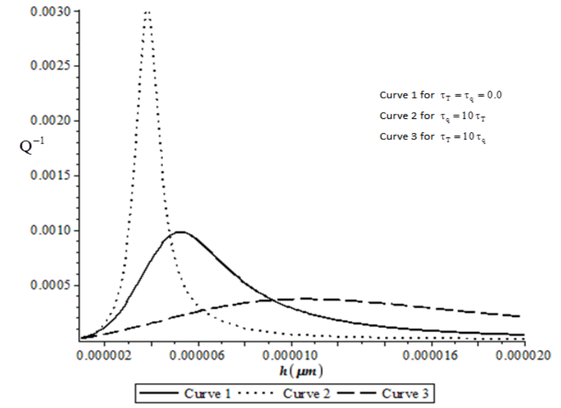


**Figure 3:** The thermoelastic damping of clamped beam with length $L = 100\mu m$ for set I

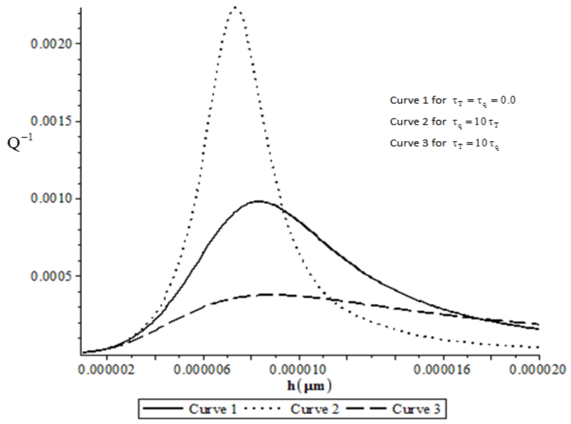


**Figure 4:** The thermoelastic damping of clamped beam with length $L = 100\mu m$ for set II

In Figures 5-8, we represent the variation of frequency shift with varying thickness $\left(10^{-9} \le h \le 10^{-4}\right)$ and fixed length $L = 500\mu m$, for the set I and set II. It is observed that the frequency shift increases with decreasing thickness and then it becomes stable for some values of thickness, then it decreases rapidly in the considered range of thickness (h).

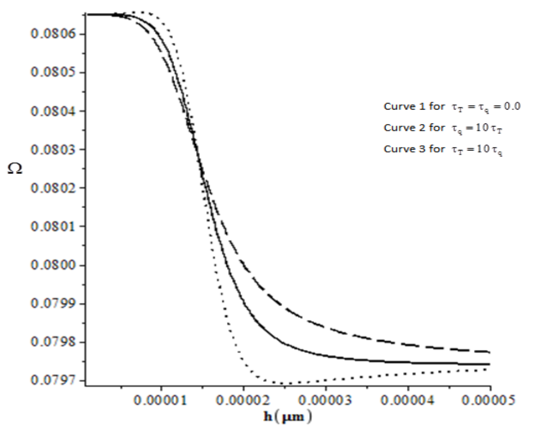


**Figure 5:** The frequency shift of clamped beam with length $L = 500\mu m$ for set I

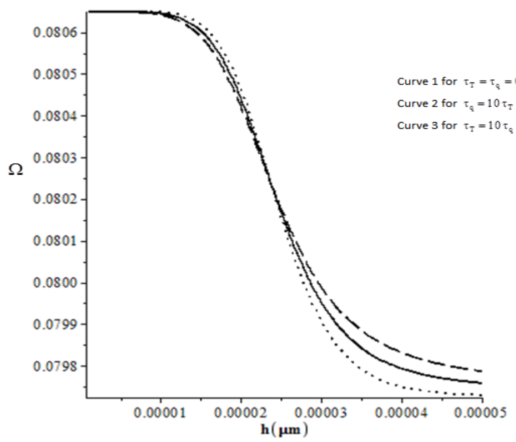


**Figure 6**: The frequency shift of clamped beam with length $L = 500\mu m$ for set II

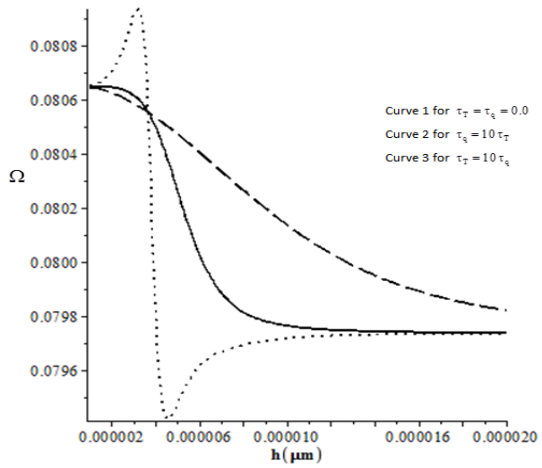


**Figure 7:** The frequency shift of clamped beam with length $L = 100\mu m$ for set I

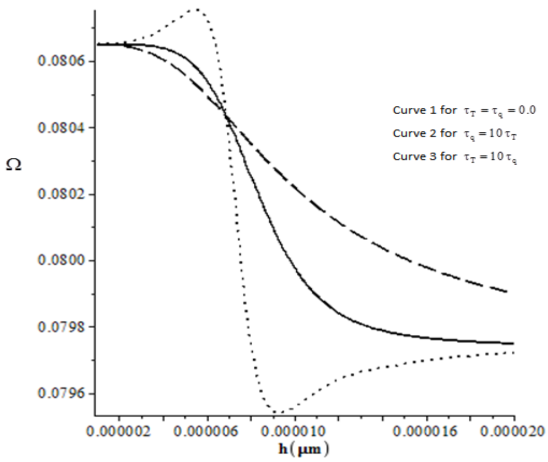


**Figure 8:** The frequency shift of clamped beam with length $L = 100\mu m$ for set II

It is observed in Figs. 7, 8 the variation of the thermoelastic frequency shift has not any peak points in the cases of $\tau_T = \tau_q = 0$ and $\tau_q = 10\tau_T$ while it has two peak points in the case of $\tau_T = 10\tau_q$ for set I and set II and the position of that peak points have been shifted through h length.

In Figures 9-12, we represent the variation of thermoelastic damping (scaled by the relaxation strength $\Delta_E$) and the variation of frequency shift, respectively, with varying length and fixed $h = 5\mu m$, for the set I and set II.

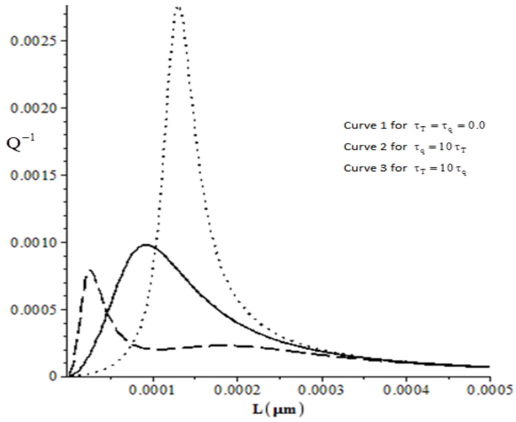


**Figure 9:** The thermoelastic damping of clamped beam with length $h = 5\mu m$ for set I

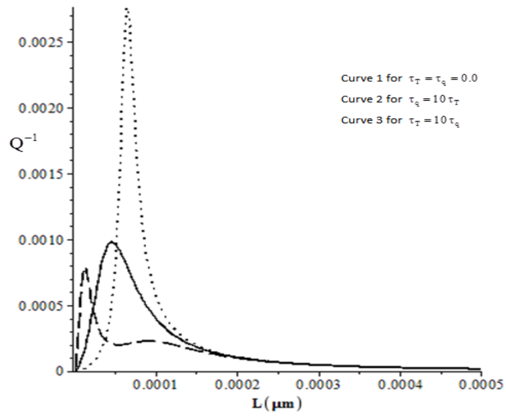


**Figure 10:** The thermoelastic damping of clamped beam with length $h = 5\mu m$ for set II

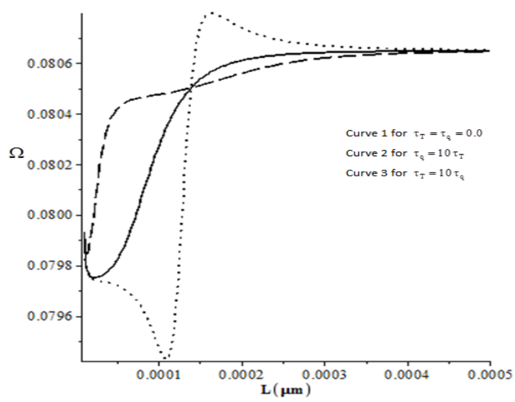


**Figure 11:** The frequency shift of clamped beam with length $h = 5\mu m$ for set I

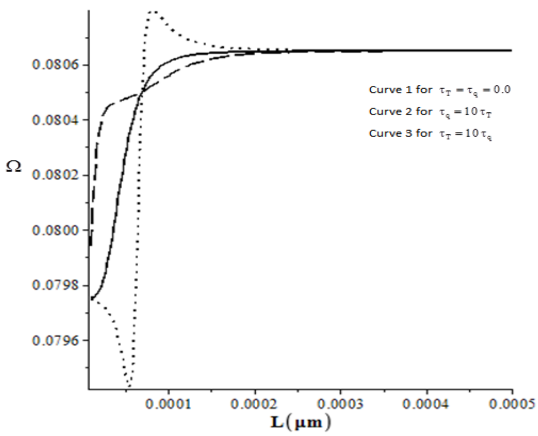


**Figure 12:** The frequency shift of clamped beam with length $h = 5\mu m$ for set II

It is observed that damping factor increases first then decreases in the considered range of length (L). Also, It is observed the value of damping factor in curve 2 is greatest value than curve 1 and 3. Also, we observed in Fig(11, 12) that the frequency shift increases rapidly with decreasing thickness until it attains to a maximum value and then it becomes stable for small values of length.

## Conclusion

It is observed that magnitude of the peak value of damping increases with adding part of voids and thermal relaxation time. In addition to decreasing the thickness and length of the beam, it is observed that thickness and length of the beam decrease with increasing values of modes in three models beam resonators. It is observed that the frequency shift increases with decreasing thickness and then it becomes stable for some values of thickness, then it decreases rapidly in the considered range of thickness.

## Appendix

The quantities $p_0, q_0, \theta_1, \theta_2$ and $\theta_3$ in eqs. (48-49) are defined as

$$p_0 = \sqrt{\frac{\rho C_\upsilon \omega_0 r_1 r_2}{2K}}, \qquad r_1 = \sqrt{R_0^2 + S_0^2},$$

$$r_2 = \sqrt{1 + \left(\bar{\tau}_q / \bar{\tau}_T\right)^2 \omega_0^2}, \tag{A.1}$$

$$\theta_1 = \tan^{-1}\left(\frac{S_0}{R_0}\right), \qquad \theta_2 = \tan^{-1}\left(-\frac{1}{\left(\bar{\tau}_q / \bar{\tau}_T\right)\omega_0}\right) \tag{A.2}$$

$$R_0 = \frac{\beta\left(\rho\chi\omega_0^2 - \xi_1\right)^2 + mb\left(\rho\chi\omega_0^2 - \xi_1\right) + \beta\omega_0^2\xi_2^2}{\beta\left[\left(\rho\chi\omega_0^2 - \xi_1\right)^2 + \omega_0^2\xi_2^2\right]},$$

$$S_0 = \frac{\omega_0\xi_2\left(mb - (\beta - 1)\left(\rho\chi\omega_0^2 - \xi_1\right)\right)}{\beta\left[\left(\rho\chi\omega_0^2 - \xi_1\right)^2 + \omega_0^2\xi_2^2\right]},$$

$$q_0 = \sqrt{\frac{1}{2\alpha}\left(1 + \frac{m\beta T_0}{b\rho C_\upsilon}\right) r_3}, \qquad r_3 = \sqrt{\left(\rho\chi\omega_0^2 - \xi_1\right)^2 + \omega_0^2 \xi_2^2}$$

$$\theta_3 = \tan^{-1}\left(-\frac{\omega_0\xi_2}{\rho\chi\omega_0^2 - \xi_1}\right) \tag{A.3}$$

The quantities $H_1, H_2, H_3, H_4$ in eqs.(51-52) are given by

$$H_1 = 1 + \frac{6}{\hat{\kappa}^2}\cos\theta_3 - \frac{6\sqrt{2}\cos\left(3\theta_3/2\right)}{\hat{\kappa}^3}\left(\frac{\sin\hat{\kappa}' + \tan\left(3\theta_3/2\right)\sinh\hat{\kappa}'\hat{T}^*}{\cos\hat{\kappa}' + \cosh\hat{\kappa}'\hat{T}^*}\right) \tag{A.4}$$

$$H_2 = \left\{1 + \frac{6}{\kappa^2}\cos\left(\theta_1 + \theta_2\right) - \frac{6\sqrt{2}\cos\left(\left(3\left(\theta_1 + \theta_2\right)\right)/2\right)}{\kappa^3}\left(\frac{\sin\kappa' + \tan\left(\left(3\left(\theta_1 + \theta_2\right)\right)/2\right)\sinh\kappa' T^*}{\cos\kappa' + \cosh\kappa' T^*}\right)\right\} \tag{A.5}$$

$$H_3 = \left\{-\frac{6}{\hat{\kappa}^2}\sin\theta_3 - \frac{6\sqrt{2}\cos\left(3\theta_3/2\right)}{\hat{\kappa}^3}\left(\frac{\sinh\hat{\kappa}'\hat{T}^* - \tan\left(3\theta_3/2\right)\sin\hat{\kappa}'}{\cos\hat{\kappa}' + \cosh\hat{\kappa}'\hat{T}^*}\right)\right\} \tag{A.6}$$

$$H_4 = \left\{-\frac{6}{\kappa^2}\sin\left(\theta_1 + \theta_2\right) - \frac{6\sqrt{2}\cos\left(\left(3\left(\theta_1 + \theta_2\right)\right)/2\right)}{\kappa^3}\left(\frac{\sinh\kappa' T^* - \tan\left(\left(3\left(\theta_1 + \theta_2\right)\right)/2\right)\sin\kappa'}{\cos\kappa' + \cosh\kappa' T^*}\right)\right\} \tag{A.7}$$

$$\kappa = p_0^* h, \qquad \hat{\kappa} = q_0^* h \tag{A.8}$$

$$\kappa' = \sqrt{2}\kappa\cos\left(\frac{\theta_1 + \theta_2}{2}\right), \qquad \hat{\kappa}' = \sqrt{2}\hat{\kappa}\cos\left(\frac{\theta_3}{2}\right),$$

$$T^* = \tan\left(\frac{\theta_1 + \theta_2}{2}\right), \qquad \hat{T}^* = \tan\left(\frac{\theta_3}{2}\right) \tag{A.9}$$

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