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Rational Frieze Sequences Associated to a 2x2 Generalized Kronecker Quiver

Dimitris Karayannakis [a] *, Joseph Serafimides* & Dionysios Kladis**

Deparment of Informatics Engineering,TEI of Crete, Greece,([a]Subsector of Mathematics and *Programming Algorithms' Development Lab)
Deparment of Informatics Engineering,TEI of Crete, Greece,(**Multimedia Content Lab)

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Abstract

We obtain, in closed form, a family of frieze sequences that corresponds to a certain type vertex labeling of a generalized version of the classical 2x2 Kronecker quiver. We also calculate explicitly, for the obtained family of sequences, a rational "PC friendly" subfamily sample, via Mathematica .

Keywords: *Frieze, Quiver, Generalized Fibonacci, Linear Recursive Sequence*

1. Introduction

A quiver is a directed acyclic (possibly multi-edged) graph. When a quiver is given, under a specific and suitable initial condition , a labeling of its vertices can be recursively defined thus leading to a so called frieze associated to the quiver which is a unique (due to the acyclicity) sequence of labels .For simplicity we will call it a frieze sequence (for a given particular quiver).

A classical example is that of the 2x2 Kronecker quiver (i.e. two vertices and two edges from one to another): if V is the set of its vertices, starting with the labeling (v,0)=v(0)→(v,1)=v(1) e.t.c., for each vertex $v\in V$, we obtain as a frieze sequence the even rank Fibonacci numbers (see e.g. [1]). A more general frieze can be produced when v(0) is taken to be a variable and then with the first two labels v(0) and v(1) taken to be a and b, respectively, this 2x2 Kronecker quiver is associated with the frieze sequence defined through the recursive formula $u_{n+2}=z(a,b)u_{n+1}-u_n$ for $u_0=a$ and $u_1=b$, $ab\neq 0$, which evidently generalizes the recursion of the even rank Fibonacci numbers.

It has been proved that for z (a, b) = $(a^2+b^2+1)/ab$, $ab\neq 0$, (see e.g. [2])

$$u_n=\frac{1}{a^{n-1}b^{n-2}}(1,b)\,M^{n-2}\begin{pmatrix}1\\ b\end{pmatrix},\ n\geq 2 \text{ with } M=\begin{pmatrix}a^2+1 & b\\ b & b^2\end{pmatrix} \quad \textbf{(0.1)}$$

2. Main description and closed form calculations

Let M= (a_{ij}) be any 2x2 matrix. An elementary and direct use of the Cayley-Hamilton theorem gives us the formula

$$M^2-(\mathrm{tr}M)\,M+|M|\,I=O \quad \textbf{(1.1)}$$

where trM and |M| indicate, respectively, the trace and the determinant of M and O the 2x2 zero matrix. In particular, for $a_{11}=a^2+1$, $a_{12}=a_{21}=b$ and $a_{22}=b^2$, with a, b real numbers, by repeated use of (1), we obtain with the evident abuse of notation

$$M^n=\omega_n M-|M|\,\omega_{n-1} \text{ for } n\geq 1 \text{ with } \omega_0=0,\ \omega_1=1 \quad \textbf{(1.2)}$$

where $\omega_{n+1}=(a^2+b^2+1)\,\omega_n-(ab)^2\,\omega_{n-1}$

The classical theory for recursive sequences of the form $\omega_{n+1}=c_1\,\omega_n+c_2\,\omega_{n-1}$ (e.g. see [3]) leads to the expression

$$\omega_n=A\lambda_1^n+B\lambda_2^n, \quad \textbf{(1.3)}$$

where A, B are arbitrary constants and λ_1, λ_2 the roots of the equation $\lambda^2-(a^2+b^2+1)\lambda+(ab)^2=0$. Note that (1.3) is the appropriate formula here since the discriminant Δ is nonzero (in fact $\Delta\geq 1$).

For $\omega_0=0$ and $\omega_1=1$ we obtain also that $A=1/(\lambda_1-\lambda_2)$ and B=-A. We conclude that

$$\omega_n=\frac{(ab)^{n-1}}{2^n\sqrt{\gamma}}\{(z+\sqrt{\gamma})^n-(z-\sqrt{\gamma})^n\}, \quad \textbf{(1.4)}$$

where we have set $z=(a^2+b^2+1)/ab$ and $\gamma=z^2-4$.Note also that |Z|>2 and due to symmetry ,in the rest of our work we will focus only upon the case a>0 ,b>0, a domain where clearly z=z(a,b) lacks minimum but it has infimum=2.

It is now evident that when (1.2) is combined with (1.4) we have a closed form description of M^n in terms of M, for any given n and any real pair a, b, and thus using (0.1), we have a computer friendly formula to work with that can provide the frieze sequence u_n.

3. Ramifications

For reasons that will immediately become clear in the calculations that follow, we parameterize the initial terms u_0=a and u_1=b (and thus sequences u_n, ω_n and the matrix powers M^n) via a= $\frac{p^2+1}{p^2-1}$ and b= $\frac{2p}{p^2-1}$, with p>1. In paragraph 4 we limit ourselves to rational values of p and we provide the image of the surface mesh z=z(a,b) using mainly a sample of rational points in 3D (Appendix A). For this parameterization z= $\frac{p^2+1}{p}$, $\sqrt{\gamma}=\frac{p^2-1}{p}$ and now (1.4) can be put in to an even more "PC friendly" form:

$$\omega_n = \frac{(2p^2+2)^{n-1}(p^{2n}-1)}{(p^2-1)^{2n-1}} \quad \textbf{(2.1)}$$

We then conclude that, for n ≥ 3,

$$u_n = (p^{2n-2}+1)/(p^2-1)\ p^{n-2} \quad \textbf{(2.2)}$$

Remarks:

1. Note that for n=2, as an immediate result of (0.1) combined with our parameterization that leads to $a^2=b^2+1$, we obtain u_2=a.

2. One could, evidently, combine the outcome of paragraph 2 and establish a rather cumbersome formula for the sum of the first N terms of $\{u_n\}$.In the frame of the above particular parametric formulation though the sum is simple and we can easily check that, for N≥3,

$$\sum_{n=0}^{N} u_n = \frac{p^{2N-1}+2p^N-p^{N-1}-1}{p^{N-2}(p^2-1)(p-1)} \quad \textbf{(2.3)}$$

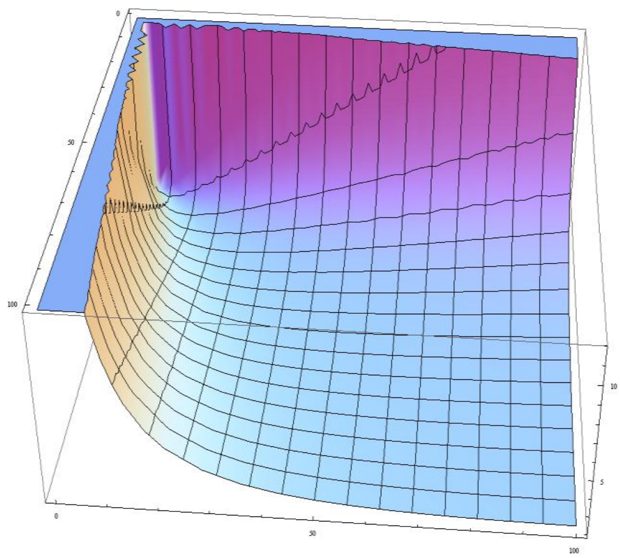
4. Numerical (rational) calculations via Mathematica (Tables 1, 2, Appendices A,B)

p	u_0=a	u_1=b	z(a, b)
11/10	221/21	220/21	221/110
6/5	61/11	60/11	61/30
13/10	269/69	260/69	269/130
7/5	37/12	35/12	74/35
3/2	13/5	12/5	13/6
8/5	89/39	80/39	89/40
17/10	389/189	340/189	389/170
9/5	53/28	45/28	106/45
19/10	461/261	380/261	461/190
2	5/3	4/3	5/2
21/10	541/341	420/341	541/210
11/5	73/48	55/48	146/55
23/10	629/429	460/429	629/230
12/5	169/119	120/119	169/60
5/2	29/21	20/21	29/10
13/5	97/72	65/72	194/65
27/10	829/629	540/629	829/270
14/5	221/171	140/171	221/70
29/10	941/741	580/741	941/290
3	5/4	3/4	10/3
31/10	1061/861	620/861	1061/310
16/5	281/231	160/231	281/80
33/10	1189/989	660/989	1189/330
17/5	157/132	85/132	314/85
7/2	53/45	28/45	53/14
18/5	349/299	180/299	349/90
37/10	1469/1269	740/1269	1469/370
19/5	193/168	95/168	386/95
39/10	1621/1421	780/1421	1621/390
4	17/15	8/15	17/4

Table 2: indicative u_n for p=11/10 (rounding up for n>15)

n	$u_n = \frac{1}{a^{n-1}b^{n-2}}(1\ b)M^{n-2}\begin{pmatrix}1\\ b\end{pmatrix}$ =$(p^{2n-2}+1)/(p^2-1)p^{n-2}$
2	$\frac{522422872400}{4084101}$
3	$\frac{30707204213842}{224625555}$
5	$\frac{202124980347430361}{1358984607750}$
10	$\frac{4464201682802640772535710961}{21886583006274525000000}$
15	0.306412X106
20	0.479768X106
25	0.764157X106
30	1.2254X106
40	3.17104X106
50	8.22203X106

Appendix A: Plot of $z = \frac{x^2+y^2+1}{xy}$, x=a, y=b



Appendix B: Calculations via Mathematica 8

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For[n=2,n<15,n++,
 p=11/10;
 m=n-1;
 a=((p^2+1)/(p^2-1));
 b=(2*p/(p^2-1));
 wnk=2-1+n (-1+p^2)1-2 *n (1+p^2)-1+n (-1+ p2*n);
 wnpk=2-1+m (-1+p^2)1-2 *m (1+p^2)-1+m (-1+ p2*m);
 mat={{a^2+1,b},{b,b^2}};
 mati={{1,0},{0,1}};
 mn=wnk*mat-(a*b)^2*wnpk*mati;
 Print[n ," ",wnk," ",mn //N];
 Print[(1/(a^(n-1)*b^(n-2)))*({1,b}.mn.{{1},{b}})];
 Print[(1/(a^(n-1)*b^(n-2)))*({1,b}.MatrixPower[mat,n].{{1},{b}})];]
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