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Common fixed points of generalized Geraghtyco-cyclic contractive maps

M. Rajani* and V. A. Kumari

Department of Mathematics, Institute of Aeronautical Engineering, Dundigal, Hyderabad - 500 043, A. P., India

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Abstract

In this paper, we introduce generalized Geraghty co-cyclic contractive maps and prove existence of common fixed point results in complete metric spaces. We deduce some corollaries from our main results and provide examples in support of our results.

Keywords: *Cyclic Representation, Co-Cyclic Representation, Geraghty Co-Cyclic Contractive Map, Generalized Geraghty Co-Cyclic Contractive Maps.*

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1. Introduction

In 1973, Geraghty**[7]**introduced an extension of the contraction in which the contraction constant was replaced by a function having some specified properties. We use the following notation introduced by Geraghty, namely

$S=\{\beta:[0,\infty)\to[0,1)/\,\beta(t_n)\to 1\Rightarrow t_n\to 0\}$.

Definition 1.1.(Geraghty [7])A selfmap $f:X\to X$ is said to be a Geraghty contraction if there exists $\beta\in S$ such that for all $x,y\in X$

$$d(fx,fy)\le\beta(d(x,y))d(x,y) \qquad (1.1.1)$$

Theorem 1.2.(Geraghty [7])Let X be a complete metric space. Let $f:X\to X$ be a mapping such that there exists $\beta\in S$ such that for all $x,y\in X$

$$d(fx,fy)\le\beta(d(x,y))d(x,y) \ (1.2.1)$$

Then for any choice of initial point x_0 the iteration $x_n=f(x_{n-1})$ for $n=1,2,3,....$, converges to the unique fixed point z of f in X.

In 1997, Alber and Guerre-Delabriere[2] introduced weakly contractive mappings as a generalization of contraction maps and proved some fixed point results in Hilbert space setting. In 2001, Rhoades [10] extended this concept to Banach spaces.

In 2003, Kirk, Srinivasan and Veeramani[9] introduced cyclic contractions and proved fixed point results for not necessarily continuous mappings.

*Corresponding author's ORCID ID: 0000-0000-0000-0000
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In 2013, Harjani, Lopez and Sadarangani[6] proved existence of fixed points of continuous cyclic weakly contractive selfmaps in complete metric spaces. Recently, Alemayehu [1] introduced co-cyclic weakly contractive maps and proved common fixed points results in compact metric spaces. We denote

$\tau=\{\varphi:[0,\infty)\to[0,\infty)/\})\,/\,\varphi\, is\, non-decreasing,$
$\varphi(0)=0,\varphi(t)>0\; for\, t>0\}$.

Definition 1.3. [`11]Let X be a non-empty set, m a positive integer and $f:X\to X$ a selfmap and $X=\cup_{i=1}^{m}A_i$ is said to be a cyclic representation of X with respect to the map f if (i) A_i, i= $1,2,...,m$ are non-empty subsets of X

(ii) $f(A_1)\subset A_2,....,f(A_{m-1})\subset A_m, f(A_m)\subset A_1$.

Definition 1.4. [1]Let X be a non-empty set, m a positive integer and $T,f:X\to X$ a selfmap and $X=\cup_{i=1}^{m}A_i$ is said to be a co-cyclic representation of X w. r. t. T,f if

(i) A_i, i= $1,2,...,m$ are non-empty subsets of X

(ii) $T(A_1)\subset fA_2,....,T(A_{m-1})\subset fA_m, T(A_m)\subset fA_1$.

Here we note that, by taking f as the identity map, we get a cyclic representation of X with respect to the selfmap T introduced by Rus[11]

Definition 1.5. [1 Let (X,d) be a non-empty set, m a positive integer, $A_1,A_2,...,A_m$ are closed non-empty subsets of X and $X=\cup_{i=1}^{m}A_i$. Let $T,f:X\to X$ be two T is said to be a co-cyclic weakly contractive map w. r. t. f if

(i) $X = \cup_{i=1}^{m} A_i$ is said to be a co-cyclic representation of X w. r. t. T and f.

(ii) $d(Tx, Ty) \le d(fx, fy) - \varphi(d(fx, fy))$ (1.5.1)

for any $x \in A_i$ and $y \in A_{i+1}$, where $A_{m+1} = A_1$ and $\varphi \in \tau$.

Definition 1.6. [8]Two self-mappings f and T of a metric space (X, d) are said to be weakly compatible if they commute at their coincidence points, i.e., if $fu = Tu$ for $u \in X$, then $fTu = fTu$.

Remark 1.7. In [1] maps f, T satisfying (i) and (ii) of Definition 1.5 are mentioned as `co-cyclic weak contractions'. But the terminology `T co-cyclic weakly contractive map w. r. t. f' is more appropriate as the inequality (1.5.1) is indicating `weakly contractive' property. For more details, we refer [2] ,[10], [5] and [3].

Theorem 1.8. [4]Let (X, d) be a compact metric space and let $T, f : X \to X$ be two selfmaps. Suppose that m a positive integer, $A_1, A_2, ..., A_m$ are closed non-empty subsets of X and $X = \cup_{i=1}^{m} A_i$. Let $T, f : X \to X$ be two T is said to be a co-cyclic weakly contractive map w. r. t. f. If f is one-one and T and f are continuous, then f and T have a coincidence point in X. Further, if the maps f and T are weakly compatible then f and T have a unique common fixed point in X.

Let f and T be two selfmaps of a metric spaces (X, d) In Section 2, we define Geraghty co-cyclic contractive maps T w. r. t. f by using a function $\beta \in S$ and prove the existence of common fixed points in complete metric spaces. In Section 3, we define generalized Geraghty co-cyclic contractive maps T w. r. t. f by using $\beta \in S$ and prove the existence of common fixed points in complete metric spaces. In Section 4, we deduce some corollaries from our main results and provide examples in support of our results.

2. Common fixed points of Geraghty co-cyclic contractive maps

In the following, we introduce Geraghty co-cyclic contractive maps by using an element $\beta \in S$.

Definition 2.1. Let (X, d) be a non-empty set, m a positive integer, $A_1, A_2, ..., A_m$ are closed non-empty subsets of X and $X = \cup_{i=1}^{m} A_i$. Let $T, f : X \to X$ be two T is said to be Geraghty co-cyclic contractive map w. r. t. f if

(i) $X = \cup_{i=1}^{m} A_i$ is said to be a co-cyclic representation of X w. r. t. T and f.

(ii) there exists $\beta \in S$ such that

$$d(Tx, Ty) \le \beta(d(fx, fy))d(fx, fy)$$

(2.1.1)

for any $x \in A_i$ and $y \in A_{i+1}$, where $A_{m+1} = A_1$.

Theorem2.2. Let (X, d) be a complete metric space and Let $T, f : X \to X$ be two selfmaps. Suppose that m a positive integer, $A_1, A_2, ..., A_m$ are closed non-empty subsets of X and $X = \cup_{i=1}^{m} A_i$ and T is said to be Geraghty co-cyclic contractive map w. r. t. f if f is one-one and $f(A_i)$ is closed, then there exists $z \in \cap_{i=1}^{m} A_i$ such that z is a coincidence point of f and T.

Proof: Let $x_0 \in X = \cup_{i=1}^{m} A_i$. Then $x_0 \in A_i$ for some $i \in \{1, 2, 3, .., m\}$. Then $Tx_0 \in T(A_i) \subset f(A_{i+1})$ and hence $Tx_0 = fx_1 \in f(A_{i+1})$ for some $x_1 \in A_{i+1}$.

Now, since $Tx_1 \in T(A_{i+1}) \subset f(A_{i+2}),$ we have $Tx_1 = fx_2$ for some $x_2 \in A_{i+2}$.

On continuing this process, we get a sequence $\{x_n\} \subset X$ such that

$Tx_n = fx_{n+1}$ forall $n = 1, 2, ...$. (2.2.1)

Hence, for each n, there exists a positive integer $i_n \in \{1, 2, ..., m\}$ such that $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_n+1}$ satisfying $Tx_n = fx_{n+1}$ (2.2.2)

If there exists $n_0 \in \square$ with $x_{n_0} = x_{n_0+1}$, then we have $Tx_{n_0+1} = Tx_{n_0} = fx_{n_0+1}$ so that f and T have a coincidence point x_{n_0+1}.

Hence, w. l. g., we assume that $x_n \ne x_{n+1}$ for all $n = 1, 2, ...$. Then $fx_n \ne fx_{n+1}$ for all n. Further, from the construction of $\{x_n\}$, we have $Tx_n \ne Tx_{n+1}$ for all $n = 1, 2, ...$.

Now, by (2.2.2) and the inequality (2.1.1), we have

$$d(fx_n, fx_{n+1}) = d(Tx_{n-1}, Tx_n)$$
$$\le \beta(d(fx_{n-1}, fx_n))d(fx_{n-1}, fx_n) \quad (2.1.3)$$

for each $n = 1, 2, ...$. Therefore

$d(fx_n, fx_{n+1}) < d(fx_{n-1}, fx_n)$ for all $n \ge 1$.

Hence $\{d(fx_n, fx_{n+1})\}$ is a decreasing sequence of non-negative reals and hence converges to a limit r (say) , $r \ge 0$

We now show that $r = 0$.

Suppose that $r > 0$. Then from (2.2.3), we have

$d(fx_n, fx_{n+1}) \le \beta(d(fx_{n-1}, fx_n))d(fx_{n-1}, fx_n)$.

On letting $n \to \infty$, we get

$$\lim_{n\to\infty}\frac{d(fx_n,fx_{n+1})}{d(fx_{n-1},fx_n)}\le\lim_{n\to\infty}\beta(d(fx_{n-1},fx_n))\le 1$$

$1\le\lim_{n\to\infty}\beta(d(fx_{n-1},fx_n))\le 1$ sothat

$\beta(d(fx_{n-1},fx_n))\to 1,$ as $n\to\infty$.

Since $\beta\in S$, it is follows that $\lim_{n\to\infty}d(fx_{n-1},fx_n)=0.$

$r=\lim_{n\to\infty}d(fx_{n-1},fx_n)=0<r$, a contradiction.

Hence $\lim_{n\to\infty}d(fx_n,fx_{n+1})=0.\ i.e.,r=0.$

We prove that $\{fx_n\}$ is a Cauchy sequence in X.

First, we show that for every $\epsilon>0$ there exists $n\in\mathbb{N}$ such that if $p,q\ge n$ with $p-q\equiv 1(\mathrm{mod}\, m)$, then $d(fx_p,fx_q)<\epsilon.$

If it is false, then there exists an $\epsilon>0$ such that for each $n\in\mathbb{N}$ we can find sequences $\{p_n\}$ and $\{q_n\}$ such that $p_n>q_n\ge n$ with $p_n-q_n\equiv 1(mod m)$ and $d(fx_{p_n},fx_{q_n})\ge\epsilon$.

Now, let n be such that $n>2m$. Then for $q_n\ge n$ we choose $\{p_n\}$ such that $\{p_n\}$ is the smallest positive integer greater than $\{q_n\}$ satisfying $p_n-q_n\equiv 1(mod m)$ and $d(fx_{q_n},fx_{p_n})\ge\epsilon$, which implies that $d(fx_{q_n},fx_{p_{n-m}})<\epsilon$.

By using the triangular inequality, we have

$$\epsilon\le d(fx_{q_n},fx_{p_n})\le d(fx_{q_n},fx_{p_{n-m}})+\Sigma_{i=1}^{m}d(fx_{p_{n-i}},fx_{p_{n-i+1}})$$
$$<\epsilon+\Sigma_{i=1}^{m}d(fx_{p_{n-i}},fx_{p_{n-i+1}})$$

On letting $n\to\infty$, we have

$\lim_{n\to\infty}d(fx_{q_n},fx_{p_n})=\epsilon,$ since $\lim_{n\to\infty}d(fx_n,fx_{n+1})=0.$ (2.2.4)

Again, by the triangular inequality, we have

$$\epsilon\le d(fx_{q_n},fx_{p_n})\le d(fx_{q_n},fx_{q_{n+1}})+d(fx_{q_{n+1}},fx_{p_{n+1}})$$
$$+d(fx_{p_{n+1}},fx_{p_n})\le d(fx_{q_n},fx_{q_{n+1}})+d(fx_{q_{n+1}},fx_{q_n})$$
$$+d(fx_{q_n},fx_{p_n})+d(fx_{p_n},fx_{p_{n+1}})+d(fx_{p_{n+1}},fx_{p_n})$$
$$\le 2d(fx_{q_n},fx_{q_{n+1}})+d(fx_{q_n},fx_{p_n})+2d(fx_{p_{n+1}},fx_{p_n})$$

On letting $n\to\infty$ and by using (2.2.4), we have

$\lim_{n\to\infty}d(fx_{q_n+1},fx_{p_n+1})=\epsilon,$

since $\lim_{n\to\infty}d(fx_n,fx_{n+1})=0$ (2.2.5)

In fact, x_{q_n} and x_{p_n} lie in different adjacently labelled sets A_i and A_{i+1}, for $1\le i\le m$. Now by using condition (2.1.1), we have

$$d(fx_{q_{n+1}},fx_{p_{n+1}})=d(Tx_{q_n},Tx_{p_n})$$
$$\le\beta(d(fx_{q_n},fx_{p_n}))d(fx_{q_n},fx_{p_n})\quad(2.2.6)$$

On letting $n\to\infty$, using property of β in (2.2.6) and using (2.2.4) we have

$1=\frac{\epsilon}{\epsilon}\le\lim_{n\to\infty}\beta(d(fx_{q_n},fx_{p_n}))$. It is a contradiction.

So we conclude that our assumption is wrong. Therefore given $\epsilon>0$ and $n_0\in\mathbb{N}$ such that if $p,q\ge n_0$ with $p-q\equiv 1(mod m)$ then $d(fx_p,fx_q)\le\frac{\epsilon}{2}$. (2.2.7)

Since $\lim_{n\to\infty}d(fx_n,fx_{n+1})=0$, there exists $n_1\in\mathbb{N}$ such that $d(fx_n,fx_{n+1})\le\frac{\epsilon}{2m}$. (2.2.8)

for each $n\ge n_1$.

Suppose that $r,s\ge\max\{n_0,n_1\}$ and $s>r$. Then there exists $k\in\{1,2,...,m\}$ such that $s-r\equiv k(\mathrm{mod}\, m)$. We choose $j=m-k+1$. Then, since $m+1\equiv 1(\mathrm{mod}\, m)$, we have $s+j-r=s+(m-k+1)-r=(s-r)+(m+1)-k\equiv 1(\mathrm{mod}\, m)$.

$$d(fx_r,fx_s)\le d(fx_r,fx_{s+j})+d(fx_{s+j},fx_{s+j-1})$$
$$+.....+d(fx_{s-1},fx_s)$$
$$d(fx_r,fx_s)\le\frac{\epsilon}{2}+(j+1).\frac{\epsilon}{2}\le\frac{\epsilon}{2}+m.\frac{\epsilon}{2m}=\epsilon.$$

Therefore, given $\epsilon>0$ there exists $n\in\mathbb{N}$ such that $d(fx_r,fx_s)\le\epsilon$ for all $r,s\ge n$. Hence $\{fx_n\}$ is a Cauchy sequence. Since (X,d) is complete $\lim_{n\to\infty}fx_n=x$, for some $x\in X$.

Since $x_0\in X=\cup_{i=1}^{m}A_i$ implies $x_0\in A_i$ for some i and $x_l\in A_{i+l}$ for all $l\in\{1,2,...,m\}$. In particular, $x_m\in A_{i+m}=A_i$ and $x_{2m}\in A_i,....,x_{km}\in A_i$ for all $k=0,1,2,...$ .Since $\{x_{km}\}\subset A_i$, we have $\{f(x_{km})\}\subset f(A_i)$. Since $f(A_i)$ is closed and $\{f(x_{km})\}$ is a subsequence of $\{f(x_n)\}$ we have $fx_{km}\to x$ as $k\to\infty$ and $x\in f(A_i)$.

We now show that $x\in\cap_{i=1}^{m}f(A_i)$. We have $x_{l+km}\in A_{i+l+km}=A_{i+l}$ for all $l=1,2,...,m$ which implies that $f(x_{l+km})\in f(A_{i+l})$ for all l. So $i+l\equiv i_0(\mathrm{mod}\, m)$ for some $i_0\in\{1,2,...,m\}$. Therefore $f(x_{l+km})\in f(A_{i_0})$.

Now $l\in\{1,2,...,m\}$ implies $f(x_{l+km})\to x$ as $k\to\infty$. Since $f(A_{i_0})$ is closed, we have $x\in f(A_{i_0})$ Note that, for any $i\in\{1,2,...,m\}$ we have $\{i+l/l=1,2,...,m\}=\{1,2,...,m\}$ under congruent modulo m .Since this is true for any $l \in \{1,2,...,m\}$ it follows that $x \in \cap_{i=1}^{m} f(A_i)$.

Now, $x \in \cap_{i=1}^{m} f(A_i)$ implies $x \in f(A_i)$ for each $i = 1,2,...,m$. Since $f(A_i)$ is closed. Hence there exists $z_i \in A_i$ such that $x = fz_i$ for each $i = 1,2,...,m$. i.e.,~ $x = fz_1 = fz_2 = ... = fz_m$ for some $z_1 \in A_1, z_2 \in A_2, ..., z_m \in A_m$. Since f is one-one, we have $z_1 = z_2 = ... = z_m = z$ (say). Hence $x = fz, z \in \cap_{i=1}^{m} A_i$.

Now we prove that z is a coincidence point of f and T.

By the inequality (2.1.1) we have

$$d(fx_{l+km}, Tz) = d(Tx_{l+km-1}, Tz)$$
$$\leq \beta(d(fx_{l+km-1}, fz))d(fx_{l+km-1}, fz)$$

since $x_{l+km-1} \in A_{l+km-1}$ and $z \in A_{l+km}$.

On letting $k \to \infty$, we have

$\frac{d(x,Tz)}{d(x,Tz)} \leq \lim_{k\to\infty} \beta(d(fx_{l+km-1}, fz)) < 1$ .Hence

$1 \leq \lim_{k\to\infty} \beta(d(fx_{l+km-1}, fz)) < 1$ which implies that $\beta(d(fz,Tz)) \to 1$ as $k \to \infty$. Since $\beta \in S$, we have $fz = Tz$ .Therefore z is a coincidence point of f and T in X.

Theorem2.3. In addition to the hypothesis of Theorem 2.2, if the maps T and f are weakly compatible then T and f have a unique fixed point.

Proof: By Theorem 2.2, we have $fz = Tz = u$ (say). Since T and f are weakly compatible, we have $Tu = Tfz = fTz = fu$ implies $fu = Tu$.

Now, we prove that $Tu = u$.

Since $Tz \in X = \cup_{i=1}^{m} A_i$ implies $Tz \in A_i$ for some i and $z \in \cap_{i=1}^{m} A_i$, we have $z \in A_i$ for all $i \in \{1,2,...,m\}$ .Now, by the inequality (2.1.1) we have

$$d(Tz,TTz) \leq \beta(d(fz, fTz))d(fz, fTz)$$
$$\leq \beta(d(Tz,TTz))d(Tz,TTz)$$

so that $Tz = TTz$ and hence $fu = Tu = u$ .Therefore u is a common fixed point of f and T .We now show that $u \in \cap_{i=1}^{m} A_i$, since $fu = Tu = u$, we have $u \in A_i$ for some i .Now,

$$u \in A_i \Rightarrow Tu \in T(A_i) \Rightarrow Tu \in T(A_i) \subset f(A_{i+1})$$
$$\Rightarrow Tu = fv \in f(A_{i+1})$$

for some $v \in A_{i+1}$. Therefore $fu = fv$ for some $v \in A_{i+1}$, since f is one-one we have $u = v \in A_{i+1}$ so that $u \in A_{i+1}$. By repeating the same argument, we get $u \in \cap_{i=1}^{m} A_i$.

Uniqueness: Let y and z be two common fixed points of T and f. Then we have

$Ty = fy = y$ and $Tz = fz = z$ and $y, z \in \cap_{i=1}^{m} A_i$.

From the inequality (2.1.1), we have

$$d(y,z) = d(Ty,Tz) \leq \beta(d(fy, fz))d(fy, fz)$$
$$\leq \beta(d(y,z))d(y,z))$$

since $\beta \in S$ we have $d(y,z) = 0$. i.e., $z = y$.

Therefore f and T have a unique common fixed point in X.

Example2.4. Let $X = \mathbb{R}$ with the usual metric.Let $A_1 = (-\infty,1]$ and $A_2 = (0,\infty)$.

We define $T, f : X \to X$ by $Tx = \frac{3-x}{2}$ and $fx = 2 - x$ .We define $\beta : [0,\infty) \to [0,1)$ by $\beta(t) = \frac{1}{1+t}, t > 0$ .Clearly, $X = A_1 \cup A_2$ is co-cyclic representation of X w.r.t. T and f.

Now we verify the inequality (2.1.1) in the following:

For $x \in A_1$ and $y \in A_2$, then $d(Tx,Ty) =| \frac{x}{2} - \frac{y}{2} |$ and

$$d(fx, fy) =| x - y |$$
$$d(Tx,Ty) =| \frac{x}{2} - \frac{y}{2} |\leq \beta(| x - y |)(| x - y |)$$
$$= \beta(d(fx, fy))d(fx, fy).$$

Clearly, T and f are weakly compatible and satisfy all the hypotheses of Theorem~2.3. Therefore \$1\$ is the unique common fixed point of T and f and $1 \in A_1 \cap A_2$.

If we relax the f is not one to one of Theorem~2.3 then T and f may have a common fixed point.

Example2.5. Let $X = \{0,1,2,3\}$ with the usual metric.Let $A_1 = \{0,1\}$ and $A_2 = \{12,3\}$. We define $T, f : X \to X$ by $T0 = 1, T1 = 1, T2 = 0, T3 = 1$

$f0 = 0, f1 = 1, f2 = 3$ and $f3 = 0$ .We define

$\beta : [0,\infty) \to [0,1)$ by $\beta(t) = \begin{cases} 0 & if\ t = 0; \\ \frac{4+t}{4+2t} & if\ t > 0. \end{cases}$

Clearly, $X = A_1 \cup A_2$ is co-cyclic representation of X w.r.t. T and f.

Now we verify the inequality (2.1.1) in the following:

Case (i): $x = 0$ and $y = 2$, then $d(T0,T2) = 1$ and $d(f0, f2) = 3$.

$$d(Tx,Ty) = d(T0,T2) = 1 \leq \beta(3)3$$
$$= \beta(d(f0, f2))d(f0, f2) = \beta(d(fx, fy))d(fx, fy).$$

Case (ii): $x=1$ and $y=2$, then $d(T1,T2)=1$ and $d(f1,f2)=2$.

$d(Tx,Ty)=d(T1,T2)=1\leq\beta(2)2$

$=\beta(d(f1,f2))d(f1,f2)=\beta(d(fx,fy))d(fx,fy)$.

Case (iii): $x=0$ and $y=1$.

In this case, the inequality(2.2.1) trivially holds.

Case (iv): $x=0$ and $y=3$.

In this case, the inequality (2.2.1) trivially holds.

Case (v): $x=1$ and $y=3$.

In this case, the inequality (2.2.1) trivially holds.

Clearly, T and f are weakly compatible and satisfying all the hypotheses of Theorem2.3. Therefore 1 is the unique common fixed point of T and f and $1\in A_1\cap A_2$.

Further, we observe that at $x=1$ and $y=2$

$d(T1,T2)=1$ œ $\beta(1)1=\beta d(1,2)d(1,2)$ for any $\beta\in S$.

Hence, the inequality (1.1.1) does not hold for any β Therefore T is not a Geraghty contractive map. Hence Theorem1.2 is not applicable.

3. Common fixed points of generalized Geraghty co-cyclic contractive maps

In the following, we introduce generalized Geraghty co-cyclic contractive maps by using an element $\beta\in S$.

Definition 3.1. Let (X,d) be a non-empty set, m a positive integer, $A_1,A_2,...,A_m$ are closed non-empty subsets of X and $X=\cup_{i=1}^{m}A_i$. Let $T,f:X\to X$ be two T is said to be generalized Geraghty co-cyclic contractive map w. r. t. f if

(i) $X=\cup_{i=1}^{m}A_i$ is said to be a co-cyclic representation of X w. r. t. T and f.

(ii) there exists $\beta\in S$ such that

$d(Tx,Ty)\leq\beta(M(x,y))M(x,y)$

(3.1.1)

for any $x\in A_i$ and $y\in A_{i+1}$, where $A_{m+1}=A_1$, where

$M(x,y)=\max\{d(fx,fy),d(fx,Tx),d(fy,Ty),\frac{1}{2}(d(fx,Ty)+d(Tx,fy))\}$

Theorem3.2. Let (X,d) be a complete metric space and Let $T,f:X\to X$ be two selfmaps. Suppose that m a positive integer, $A_1,A_2,...,A_m$ are closed non-empty subsets of X and $X=\cup_{i=1}^{m}A_i$ and T is said to be generalized Geraghty co-cyclic contractive map w. r. t. f if f is one-one and $f(A_i)$ is closed, then there exists $z\in\cap_{i=1}^{m}A_i$ such that z is a coincidence point of f and T.

Proof: Let $x_0\in X=\cup_{i=1}^{m}A_i$. Then $x_0\in A_i$ for some $i\in\{1,2,3,..,m\}$. Then $Tx_0\in T(A_i)\subset f(A_{i+1})$ and hence $Tx_0=fx_1\in f(A_{i+1})$ for some $x_1\in A_{i+1}$.

Now, since $Tx_1\in T(A_{i+1})\subset f(A_{i+2})$, we have $Tx_1=fx_2$ for some $x_2\in A_{i+2}$.

On continuing this process, we get a sequence $\{x_n\}\subset X$ such that $Tx_n=fx_{n+1}$ for all $n=1,2,\ldots$.

(3.2.1)

Hence, for each n, there exists a positive integer $i_n\in\{1,2,...,m\}$ such that $x_n\in A_{i_n}$ and $x_{n+1}\in A_{i_n+1}$ satisfying $Tx_n=fx_{n+1}$. (3.2.1)

If there exists $n_0\in$ with \$ $x_{n_0}=x_{n_0+1}$, then we have $Tx_{n_0+1}=Tx_{n_0}=fx_{n_0+1}$ so that f and T have a coincidence point x_{n_0+1}.

Hence, w. l. g., we assume that $x_n\neq x_{n+1}$ for all $n=1,2,\ldots$. Then $fx_n\neq fx_{n+1}$ for all n. Further, from the construction of $\{x_n\}$, we have $Tx_n\neq Tx_{n+1}$ for all $n=1,2,\ldots$.

Now, by (3.2.2) and the inequality (3.1.1), we have

$d(fx_n,fx_{n+1})=d(Tx_{n-1},Tx_n)\leq\beta(M(x_{n-1},x_n))M(x_{n-1},x_n)$

, where (3.2.3)

$$M(x_{n-1},x_n)=\max\{d(fx_{n-1},fx_n),d(fx_{n-1},Tx_{n-1}),d(fx_n,Tx_n),\frac{1}{2}(d(fx_{n-1},Tx_n)+d(fx_n,Tx_{n-1}))\}$$

$$=\max\{d(fx_{n-1},fx_n),d(fx_{n-1},fx_n),d(fx_n,fx_{n+1}),\frac{1}{2}(d(fx_{n-1},fx_{n+1})+d(fx_n,fx_n))\}$$

$$\leq\max\{d(fx_{n-1},fx_n),d(fx_n,fx_{n+1}),\frac{1}{2}(d(fx_{n-1},fx_n)+d(fx_n,fx_{n+1}))\}$$

$$=\max\{d(fx_{n-1},fx_n),d(fx_n,fx_{n+1})\}.$$

If $\max\{d(fx_{n-1},fx_n),d(fx_n,fx_{n+1})\}=d(fx_n,fx_{n+1})$ then, from (3.2.3) we have

$d(fx_n,fx_{n+1})\leq\beta(d(fx_n,fx_{n+1}))d(fx_n,fx_{n+1})<d(fx_n,fx_{n+1})$, a contradiction.

Hence $\max\{d(fx_{n-1},fx_n),d(fx_n,fx_{n+1})\}=d(fx_{n-1},fx_n)$ then, from (3.2.3) we have

$d(fx_n,fx_{n+1})\leq\beta(d(fx_{n-1},fx_n))d(fx_{n-1},fx_n)$.

(3.2.4)

By the property of β, we have

$d(fx_n,fx_{n+1})<d(fx_{n-1},fx_n)$ for all $n\geq1$.

Hence $\{d(fx_n,fx_{n+1})\}$ is a decreasing sequence of non-negative reals and hence converges to a limit r (say), $r\geq0$

We now show that $r=0$.

Suppose that $r>0$. Then from (3.2.3), we have $d(fx_n,fx_{n+1})\le\beta(M(x_{n-1},x_n))d(fx_{n-1},fx_n)$. On letting $n\to\infty$, we get

$$\lim_{n\to\infty}\frac{d(fx_n,fx_{n+1})}{d(fx_{n-1},fx_n)}\le\lim_{n\to\infty}\beta(M(x_{n-1},x_n))\le 1$$

$1\le\lim_{n\to\infty}\beta(M(x_{n-1},x_n))\le 1$ so that $\beta(d(fx_{n-1},fx_n))\to 1$, as $n\to\infty$.

Since $\beta\in S$, it is follows that $\lim_{n\to\infty}d(fx_{n-1},fx_n)=0.$

$r=\lim_{n\to\infty}d(fx_{n-1},fx_n)=0<r$, a contradiction.

Hence $\lim_{n\to\infty}d(fx_n,fx_{n+1})=0.$ $i.e., r=0.$

We prove that $\{fx_n\}$ is a Cauchy sequence in X.

First, we show that for every $\epsilon>0$ there exists $n\in\mathbb{N}$ such that if $p,q\ge n$ with $p-q\equiv 1(\operatorname{mod} m)$, then $d(fx_p,fx_q)<\epsilon$.

If it is false, then there exists an $\epsilon>0$ such that for each $n\in\mathbb{N}$ we can find sequences $\{p_n\}$ and $\{q_n\}$ such that $p_n>q_n\ge n$ with $p_n-q_n\equiv 1(mod m)$ and $d(fx_{p_n},fx_{q_n})\ge\epsilon$.

Now, let n be such that $n>2m$. Then for $q_n\ge n$ we choose $\{p_n\}$ such that $\{p_n\}$ is the smallest positive integer greater than $\{q_n\}$ satisfying $p_n-q_n\equiv 1(mod m)$ and $d(fx_{q_n},fx_{p_n})\ge\epsilon$, which implies that $d(fx_{q_n},fx_{p_{n-m}})<\epsilon$.

By using the triangular inequality, we have

$$\epsilon\le d(fx_{q_n},fx_{p_n})\le d(fx_{q_n},fx_{p_{n-m}})+\Sigma_{i=1}^{m}d(fx_{p_{n-i}},fx_{p_{n-i+1}})$$
$$<\epsilon+\Sigma_{i=1}^{m}d(fx_{p_{n-i}},fx_{p_{n-i+1}})$$

On letting $n\to\infty$, we have

$\lim_{n\to\infty}d(fx_{q_n},fx_{p_n})=\epsilon$, since $\lim_{n\to\infty}d(fx_n,fx_{n+1})=0.$ (3.2.4)

Again, by the triangular inequality, we have

$$\epsilon\le d(fx_{q_n},fx_{p_n})\le d(fx_{q_n},fx_{q_{n+1}})+d(fx_{q_{n+1}},fx_{p_{n+1}})$$
$$+d(fx_{p_{n+1}},fx_{p_n})\le d(fx_{q_n},fx_{q_{n+1}})+d(fx_{q_{n+1}},fx_{q_n})$$
$$+d(fx_{q_n},fx_{p_n})+d(fx_{p_n},fx_{p_{n+1}})+d(fx_{p_{n+1}},fx_{p_n})$$
$$\le 2d(fx_{q_n},fx_{q_{n+1}})+d(fx_{q_n},fx_{p_n})+2d(fx_{p_{n+1}},fx_{p_n})$$

On letting $n\to\infty$ and by using (2.2.4), we have

$\lim_{n\to\infty}d(fx_{q_n+1},fx_{p_n+1})=\epsilon$, since

$$\lim_{n\to\infty}d(fx_n,fx_{n+1})=0 \qquad (3.2.5)$$

In fact, x_{q_n} and x_{p_n} lie in different adjacently labelled sets A_i and A_{i+1}, for $1\le i\le m$. Now by using condition (3.1.1), we have

$d(fx_{q_{n+1}},fx_{p_{n+1}})=d(Tx_{q_n},Tx_{p_n})\le\beta(M(x_{q_n},x_{p_n}))M(x_{q_n},x_{p_n})$,

where (3.2.6)

$$M(x_{q_n},x_{p_n})=\max\{d(fx_{q_n},fx_{p_n}),d(fx_{q_n},Tx_{q_n}),d(fx_{p_n},Tx_{p_n}),$$
$$\frac{1}{2}(d(fx_{q_n},Tx_{p_n})+d(fx_{p_n},Tx_{q_n}))\}$$
$$=\max\{d(fx_{q_n},fx_{p_n}),d(fx_{q_n},fx_{q_{n+1}}),$$
$$d(fx_{p_n},fx_{p_{n+1}}),\frac{1}{2}(d(fx_{q_n},fx_{p_{n+1}})+d(fx_{p_n},fx_{q_{n+1}}))\}$$
$$=\max\{d(fx_{q_n},fx_{p_n}),d(fx_{q_n},fx_{q_{n+1}}),d(fx_{p_n},fx_{p_{n+1}}),$$
$$\frac{1}{2}(d(fx_{q_n},fx_{p_n})+d(fx_{p_n},fx_{p_{n+1}})+d(fx_{p_n},fx_{q_n})$$
$$+d(fx_{q_n},fx_{q_{n+1}}))\}\to\epsilon$$

as $n\to\infty$. On letting $n\to\infty$, using property of β in (3.2.6) and using (3.2.4) we have

$1=\frac{\epsilon}{\epsilon}\le\lim_{n\to\infty}\beta(M(x_{q_n},x_{p_n}))$. It is a contradiction.

So we conclude that our assumption is wrong. Therefore given $\epsilon>0$ and $n_0\in\mathbb{N}$ such that if $p,q\ge n_0$ with $p-q\equiv 1(mod m)$ then $d(fx_p,fx_q)\le\frac{\epsilon}{2}$. (3.2.7)

Since $\lim_{n\to\infty}d(fx_n,fx_{n+1})=0$, there exists $n_1\in\mathbb{N}$ such that $d(fx_n,fx_{n+1})\le\frac{\epsilon}{2m}$. (3.2.8)

for each $n\ge n_1$.

Suppose that $r,s\ge\max\{n_0,n_1\}$ and $s>r$. Then there exists $k\in\{1,2,...,m\}$ such that $s-r\equiv k(\operatorname{mod} m)$. We choose $j=m-k+1$. Then, since $m+1\equiv 1(\operatorname{mod} m)$, we have $s+j-r=s+(m-k+1)-r=(s-r)+(m+1)-k\equiv 1(\operatorname{mod} m)$.

$$d(fx_r,fx_s)\le d(fx_r,fx_{s+j})+d(fx_{s+j},fx_{s+j-1})$$
$$+.....+d(fx_{s-1},fx_s)$$
$$d(fx_r,fx_s)\le\frac{\epsilon}{2}+(j+1).\frac{\epsilon}{2}$$
$$\le\frac{\epsilon}{2}+m.\frac{\epsilon}{2m}=\epsilon.$$

Therefore, given $\epsilon>0$ there exists $n\in\mathbb{N}$ such that $d(fx_r,fx_s)\le\epsilon$ for all $r,s\ge n$. Hence $\{fx_n\}$ is a Cauchy sequence. Since (X,d) is complete $\lim_{n\to\infty}fx_n=x$, for some $x\in X$.

Since $x_0 \in X = \cup_{i=1}^{m} A_i$ implies $x_0 \in A_i$ for some i and $x_l \in A_{i+l}$ for all $l \in \{1,2,...,m\}$. In particular, $x_m \in A_{i+m} = A_i$ and $x_{2m} \in A_i, ..., x_{km} \in A_i$ for all $k = 0,1,2,...$ .Since $\{x_{km}\} \subset A_i$, we have $\{f(x_{km})\} \subset f(A_i)$. Since $f(A_i)$ is closed and $\{f(x_{km})\}$ is a subsequence of $\{f(x_n)\}$ we have $fx_{km} \to x$ as $k \to \infty$ and $x \in f(A_i)$.

We now show that $x \in \cap_{i=1}^{m} f(A_i)$. We have $x_{l+km} \in A_{i+l+km} = A_{i+l}$ for all $l = 1,2,...,m$ which implies that $f(x_{l+km}) \in f(A_{i+l})$ for all l. So $i + l \equiv i_0 (\bmod m)$ for some $i_0 \in \{1,2,...,m\}$. Therefore $f(x_{l+km}) \in f(A_{i_0})$.

Now $l \in \{1,2,...,m\}$ implies $f(x_{l+km}) \to x$ as $k \to \infty$. Since $f(A_{i_0})$ is closed, we have $x \in f(A_{i_0})$ Note that, for any $i \in \{1,2,...,m\}$ we have $\{i + l / l = 1,2,...,m\} = \{1,2,...,m\}$ under congruent modulo m.

Since this is true for any $l \in \{1,2,...,m\}$ it follows that $x \in \cap_{i=1}^{m} f(A_i)$.

Now, $x \in \cap_{i=1}^{m} f(A_i)$ implies $x \in f(A_i)$ for each $i = 1,2,...,m$. Since $f(A_i)$ is closed. Hence there exists $z_i \in A_i$ such that $x = fz_i$ for each $i = 1,2,...,m$. i.e.,~ $x = fz_1 = fz_2 = ... = fz_m$ for some $z_1 \in A_1, z_2 \in A_2, ..., z_m \in A_m$. Since f is one-one, we have $z_1 = z_2 = ... = z_m = z$ (say). Hence $x = fz, z \in \cap_{i=1}^{m} A_i$.

Now we prove that z is a coincidence point of f and T.

By the inequality (3.1.1) we have

$d(fx_{l+km}, Tz) = d(Tx_{l+km-1}, Tz) \le \beta(M(x_{l+km-1}, z))M(x_{l+km-1}, z)$, where (3.2.9)

$$M(x_{n_k}, z) = \max\{d(fx_{l+km-1}, fz), d(fx_{l+km-1}, Tx_{l+km-1}), d(fz, Tz), \frac{1}{2}(d(fx_{l+km-1}, Tz) + d(fz, Tx_{l+km-1}))\}$$

$$= \max\{d(fx_{l+km-1}, fz), d(fx_{l+km-1}, fx_{l+km}), d(fz, Tz), \frac{1}{2}(d(fx_{l+km-1}, Tz) + d(fz, Tx_{l+km}))\},$$

since $x_{l+km-1} \in A_{l+km-1}$ and $z \in A_{l+km}$ .On letting $k \to \infty$, we have

$$\lim_{k\to\infty} M(x_{n_k}, z) = d(fz, Tz).$$

On letting $k \to \infty$ and (3.2.9), we have

$\frac{d(x,Tz)}{d(x,Tz)} \le \lim_{k\to\infty} \beta(d(fx_{l+km-1}, fz)) < 1$. Hence $1 \le \lim_{k\to\infty} \beta(d(fx_{l+km-1}, fz)) < 1$ which implies that $\beta(d(fz, Tz)) \to 1$ as $k \to \infty$. Since $\beta \in S$, we have $fz = Tz$.

Therefore z is a coincidence point of f and T in X.

Theorem3.3. In addition to the hypothesis of Theorem 2.2, if the maps T and f are weakly compatible then T and f have a unique fixed point.

Proof: By Theorem 2.2, we have $fz = Tz = u$ (say).

Since T and f are weakly compatible, we have

$Tu = Tfz = fTz = fu$ implies $fu = Tu$.

Now, we prove that $Tu = u$.

Since $Tz \in X = \cup_{i=1}^{m} A_i$ implies $Tz \in A_i$ for some i and $z \in \cap_{i=1}^{m} A_i$, we have

$z \in A_i$ for all $i \in \{1,2,...,m\}$ .Now, by the inequality (2.1.1) we have

$d(Tz, TTz) \le \beta(M(z, Tz))M(z, Tz)$, where (3.3.1)

$$M(z, Tz) = \max\{d(fz, fTz), d(fz, Tz), d(fTz, TTz), \frac{1}{2}(d(fz, TTz) + d(fTz, Tz))\}$$

$$= \max\{d(Tz, TTz), d(Tz, Tz), d(TTz, TTz), d(Tz, TTz)\} = d(Tz, TTz)$$

so that $Tz = TTz$ and hence $fu = Tu = u$ .Therefore u is a common fixed point of f and T .We now show that $u \in \cap_{i=1}^{m} A_i$, since $fu = Tu = u$, we have $u \in A_i$ for some i.

Now,

$u \in A_i \Rightarrow Tu \in T(A_i) \Rightarrow Tu \in T(A_i) \subset f(A_{i+1}) \Rightarrow Tu = fv \in f(A_{i+1})$

for some $v \in A_{i+1}$. Therefore $fu = fv$ for some $v \in A_{i+1}$, since f is one-one we have $u = v \in A_{i+1}$ so that $u \in A_{i+1}$. By repeating the same argument, we get $u \in \cap_{i=1}^{m} A_i$.

Uniqueness: Let y and z be two common fixed points of T and f. Then we have

$Ty = fy = y$ and $Tz = fz = z$ and $y, z \in \cap_{i=1}^{m} A_i$.

From the inequality (2.1.1), we have

$d(y, z) = d(Ty, Tz) \le \beta(M(y, z))M(y, z)$,

where (3.3.2)

$$M(y, z) = \max\{d(fy, fz), d(fy, Ty), d(fz, Tz), \frac{1}{2}(d(fy, Tz) + d(Ty, fz))\}$$

$$= \max\{d(y, z), d(y, y), d(z, z), \frac{1}{2}(d(y, z) + d(y, z))\} = d(y, z)$$

From (3.3.2) and since $\beta \in S$, we have $d(y,z)=0$. i.e., $z=y$.
Therefore f and T have a unique common fixed point in X.

Corollaries and Examples

By choosing $f=I_X$ in Theorem2.2, we have the following corollary.

Corollarie 4.1. Let (X,d) be a complete metric space. Suppose that m a positive integer, $A_1, A_2, ..., A_m$ are non-empty closed subsets of X and $X=\cup_{i=1}^m A_i$. Let $T: X \to X$ be a mapping such that
(i) $\cup_{i=1}^m A_i$ is a cyclic representation of X w. r. t. T
(ii) $d(Tx,Ty) \le \beta(d(x,y))d(x,y)$ for any $x \in A_i$ and $y \in A_{i+1}$, with $A_{m+1}=A_1$,
$\beta \in S$ and each A_i is closed.Then there exist $z \in \cap_{i=1}^m A_i$ such that $Tz=z$.

By choosing $f=I_X$ in Theorem3.2, we have the following corollary.

Corollarie 4.2. Let (X,d) be a complete metric space. Suppose that m a positive integer, $A_1, A_2, ..., A_m$ are non-empty closed subsets of X and $X=\cup_{i=1}^m A_i$. Let $T: X \to X$ be a mapping such that
(i) $\cup_{i=1}^m A_i$ is a cyclic representation of X w. r. t. T .(ii) $d(Tx,Ty) \le \beta(M(x,y))M(x,y)$,where

$$M(x,y)=\max\{d(x,y),d(x,Tx),d(y,Ty),\frac{1}{2}(d(x,Ty)+d(y,Tx))\}$$

for any $x \in A_i$ and $y \in A_{i+1}$, with $A_{m+1}=A_1$, $\beta \in S$ and each A_i is closed.Then there exist $z \in \cap_{i=1}^m A_i$ such that $Tz=z$.

In the following, we provide examples in support of the results obtained in Section 3.

If we relax the f is not one to one of Theorem 3.3 then T and f may have a common fixed point.

Example 4.3. Let $X=\{1,3,5,7\}$ with the usual metric Let $A_1=\{1,3\}$ and $A_2=\{3,5,7\}$. We define $T,f: X \to X$ by
$T1=T3=3, T5=1, T7=3; f1=1, f3=3, f5=7$ and $f7=1$.

We define the same β that are mentioned in Example 2.5.Now we verify the inequality (3.1.1) in the following:

Case (i): $x=1$ and $y=5$, then $d(T1,T5)=2$ and $M(1,5)=6$.

$$d(Tx,Ty)=d(T1,T5)=2 \le \beta(6)6$$
$$=\beta(M(1,5))M(1,5)=\beta(M(x,y))M(x,y).$$

Case (ii): $x=3$ and $y=5$, then $d(T3,T5)=2$ and $M(3,5)=6$.

$$d(Tx,Ty)=d(T3,T5)=2 \le \beta(6)6$$
$$=\beta(M(3,5))M(3,5)=\beta(M(x,y))M(x,y).$$

Case (iii): $x=1$ and $y=3$.
In this case, the inequality (3.1.1) trivially holds.
Case (iv): $x=1$ and $y=7$.
In this case, the inequality (3.1.1) trivially holds.
Case (v): $x=3$ and $y=7$.
In this case, the inequality (3.1.1) trivially holds.
Clearly, T and f are weakly compatible and satisfying all the hypotheses of Theorem 3.2. Therefore 3 is the unique common fixed point of T and f and $3 \in A_1 \cap A_2$.
Further, we observe that at $x=3$ and $y=5$
$d(T3,T5)=2$ ’ $\beta(2)2=\beta(d(3,5))d(3,5)$, for any $\beta \in S$
Hence, the inequality(1.1.1) does not hold for any β. Therefore T is not a Geraghty contractive map. Hence Theorem1.2 is not applicable.
If we relax the weakly compatibility property T and f of Theorem 3.3 then T and f may not have a common fixed point.

Example 4.4. Let $X=\{0,1,2,3\}$ with the usual metric. Let $A_1=\{0,1,2\}$ and $A_2=\{1,2,3\}$. We define $T,f: X \to X$ by
$T0=1, T1=T2=2, T3=3; f0=3, f1=2, f2=1$ and $f3=0$.

We define the same β that are mentioned in Example 2.5.Now we verify the inequality (3.1.1) in the following:

Case (i): $x=0$ and $y=1$, then $d(T0,T1)=1$ and $M(0,1)=2$.

$$d(Tx,Ty)=d(T0,T1)=1 \le \beta(2)2$$
$$=\beta(M(0,1))M(0,1)=\beta(M(x,y))M(x,y).$$

Case (ii): $x=0$ and $y=2$, then $d(T0,T2)=1$ and $M(0,2)=2$.

$$d(Tx,Ty)=d(T0,T2)=1 \le \beta(2)2$$
$$=\beta(M(0,2))M(0,2)=\beta(M(x,y))M(x,y)$$

Case (iii): $x=1$ and $y=3$, then $d(T1,T3)=1$ and $M(1,3)=3$.

$$d(Tx,Ty)=d(T1,T3)=1 \le \beta(3)3$$
$$=\beta(M(1,3))M(1,3)=\beta(M(x,y))M(x,y)$$

Case (iv): $x=2$ and $y=3$, then $d(T2,T3)=1$ and $M(2,3)=3$.

$d(Tx,Ty)=d(T2,T3)=1\leq\beta(3)3$

$=\beta(M(2,3))M(2,3)=\beta(M(x,y))M(x,y)$

Case (v): $(x,y)=(1,2)$ and $(x,y)=(2,1)$.

In this case, the inequality (2.2.1) trivially holds.

Hence 2 is the coincidence point of T and f and $1\in A_1\cap A_2$.

Here, we note that T and f are not weakly compatible, since $T1=2$ and $f1=2$ then $T(f(1))=T(2)=2$ and $f(T(1))=f(2)=1$, so that $T(f(1))\neq f(T(1))$. And we observe T and f have no common fixed points in X and $1\in A_1\cap A_2$.

Further, we observe that at $x=2$ and $y=3$

$d(T2,T3)=1$’ $\beta(1)1=\beta(d(2,3))d(2,3)$ for any $\beta\in S$

Hence, the inequality(1.1.1) does not hold for any β. Therefore T is not a Geraghty contractive map. Hence Theorem1.2 is not applicable.

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